

Flat, Projective, and Locally Free Modules under Finiteness Hypotheses

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1 The equivalence theorem

Thm. 1.1. Let R be a commutative ring and M an R -module. The following conditions are equivalent.

- (1) M is finitely generated and projective.
- (2) M is finitely presented and flat.
- (3) M is finitely presented and $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module for every prime ideal $\mathfrak{p}$.
- (4) M is finitely presented and $M_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$ -module for every maximal ideal $\mathfrak{m}$.
- (5) For every prime ideal $\mathfrak{p}$ there is an $f \notin \mathfrak{p}$ such that M_f is a finite free R_f -module.

We prove

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1) \quad \text{and} \quad (2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2).$$

Proof of (1) $\Rightarrow$ (2). Suppose M is generated by n elements and choose a surjection $\pi : R^n \twoheadrightarrow M$. Put $K = \ker \pi$. Since M is projective, the exact sequence

$$0 \longrightarrow K \longrightarrow R^n \xrightarrow{\pi} M \longrightarrow 0 \tag{1}$$

splits. Thus K is a direct summand of the finite free module R^n , hence is finitely generated. Equation (1) is therefore a finite presentation of M . Every projective module is flat. $\square$

Lem. 1.2 (Base change). Let $R \rightarrow S$ be a ring homomorphism. If M is finitely presented over R , then $M \otimes_R S$ is finitely presented over S . If M is flat over R , then $M \otimes_R S$ is flat over S .

Proof. Tensor a finite presentation $R^m \rightarrow R^n \rightarrow M \rightarrow 0$ with S . Right exactness gives

$$S^m \longrightarrow S^n \longrightarrow M \otimes_R S \longrightarrow 0.$$

For flatness, if $N' \hookrightarrow N$ is an injection of S -modules, then

$$N \otimes_S (M \otimes_R S) \cong N \otimes_R M,$$

and similarly for N' . The required injection follows from the R -flatness of M . $\square$

Lem. 1.3 (Finite flat modules over a local ring). Let $(R, \mathfrak{m})$ be a local ring. Every finitely presented flat R -module M is finite free.

Proof. Choose $x_1, \dots, x_n \in M$ whose images form a basis of the $R/\mathfrak{m}$ -vector space $M/\mathfrak{m}M$. Nakayama's lemma shows that the x_i generate M . Let

$$0 \longrightarrow K \longrightarrow R^n \xrightarrow{\pi} M \longrightarrow 0, \quad e_i \longmapsto x_i.$$

Because M is finitely presented, K is finitely generated. Tensoring with $R/\mathfrak{m}$ gives part of the long exact Tor sequence:

$$\mathrm{Tor}_1^R(M, R/\mathfrak{m}) \longrightarrow K/\mathfrak{m}K \longrightarrow (R/\mathfrak{m})^n \longrightarrow M/\mathfrak{m}M \longrightarrow 0.$$

Flatness makes the Tor term zero. The last arrow is an isomorphism by the choice of the x_i , so $K/\mathfrak{m}K = 0$. A second application of Nakayama's lemma gives $K = 0$, and hence $M \cong R^n$. $\square$

Proof of (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4). For a prime $\mathfrak{p}$, apply Lemma 1.2 to $S = R_{\mathfrak{p}}$ and then Lemma 1.3. The same argument applies to maximal ideals. $\square$

The next lemma is the technical step needed to pass projectivity through localization.

Lem. 1.4 (Localization of Hom). Let M be a finitely presented R -module, let S be a flat R -algebra, and let N be any R -module. The canonical map

$$\mathrm{Hom}_R(M, N) \otimes_R S \longrightarrow \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S) \quad (2)$$

is an isomorphism.

Proof. Choose a finite presentation $R^m \rightarrow R^n \rightarrow M \rightarrow 0$. Applying $\mathrm{Hom}_R(-, N)$ gives the left-exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow N^n \longrightarrow N^m.$$

Tensoring with the flat module S preserves exactness. On the other hand, tensor the original presentation with S and apply $\mathrm{Hom}_S(-, N \otimes_R S)$. The resulting kernel diagram has identical middle and right terms, because finite direct sums commute with both Hom and tensor product. Therefore the two kernels are canonically isomorphic, proving (2). $\square$

Proof of (3) $\Rightarrow$ (1). Finite presentation already implies finite generation. To prove projectivity, let $X \twoheadrightarrow Y$ be a surjection. We must show that

$$\mathrm{Hom}_R(M, X) \longrightarrow \mathrm{Hom}_R(M, Y) \quad (3)$$

is surjective. Surjectivity of a module map can be checked after localization at every prime. By Lemma 1.4, the localization of (3) at $\mathfrak{p}$ is

$$\mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, X_{\mathfrak{p}}) \longrightarrow \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, Y_{\mathfrak{p}}).$$

This map is surjective because $M_{\mathfrak{p}}$ is free, hence projective. Thus (3) is surjective and M is projective. $\square$

Lem. 1.5 (Vanishing near a prime). If N is a finitely generated R -module and $N_{\mathfrak{p}} = 0$, then there exists $f \notin \mathfrak{p}$ such that $N_f = 0$.

Proof. Let $z_1, \dots, z_r$ generate N . Since $z_i/1 = 0$ in $N_{\mathfrak{p}}$, choose $f_i \notin \mathfrak{p}$ with $f_i z_i = 0$. Then $f = f_1 \cdots f_r \notin \mathfrak{p}$ and $fN = 0$, so $N_f = 0$. $\square$

Proof of (4) $\Rightarrow$ (5). Fix a prime $\mathfrak{p}$ and choose a maximal ideal $\mathfrak{m} \supseteq \mathfrak{p}$. By (4), $M_{\mathfrak{m}}$ is finite free, say of rank n . Lift a basis to elements $y_1, \dots, y_n \in M$ after multiplying by denominators outside $\mathfrak{m}$, and let

$$\phi : R^n \longrightarrow M, \quad e_i \longmapsto y_i.$$

After localizing at $\mathfrak{m}$, the map $\phi_{\mathfrak{m}}$ is an isomorphism. The cokernel of ϕ is finitely generated; by Lemma 1.5, after localizing at some $f_1 \notin \mathfrak{m}$, the map ϕ_{f_1} is surjective. Since M is finitely presented, $K = \ker(\phi_{f_1})$ is finitely generated over R_{f_1} . It vanishes after further localization at $\mathfrak{m}$, so another application of Lemma 1.5 gives $f_2 \notin \mathfrak{m}$ for which $K_{f_2} = 0$. Using transitivity of localization and putting $f = f_1 f_2$, we obtain

$$M_f \cong R_f^n.$$

Since $f \notin \mathfrak{m}$, certainly $f \notin \mathfrak{p}$. □

Lem. 1.6 (A finite faithfully flat cover). If $(f_1, \dots, f_r) = R$, then

$$R \longrightarrow R' = \prod_{i=1}^r R_{f_i} \tag{4}$$

is faithfully flat.

Proof. Each R_{f_i} is flat, and a finite product of flat modules is flat. For every maximal ideal $\mathfrak{m}$, some $f_i \notin \mathfrak{m}$, so $\mathfrak{m}R_{f_i} \neq R_{f_i}$ and hence $\mathfrak{m}R' \neq R'$. This is the maximal-ideal criterion for faithful flatness. □

Proof of (5) $\Rightarrow$ (2). The principal opens $D(f)$ supplied by (5) cover $\text{Spec } R$. Quasi-compactness of $\text{Spec } R$ gives a finite subcover $D(f_1), \dots, D(f_r)$; equivalently, $(f_1, \dots, f_r) = R$. Put $R' = \prod_i R_{f_i}$. By Lemma 1.6, $R \rightarrow R'$ is faithfully flat, and

$$M \otimes_R R' \cong \prod_{i=1}^r M_{f_i}. \tag{5}$$

Each M_{f_i} is finite free over R_{f_i} . An R' -module decomposes according to the orthogonal idempotents e_i of the product ring, so (5) is finite projective, in particular finitely presented and flat, over R' . Both finite presentation and flatness descend along faithfully flat ring maps. Therefore M is finitely presented and flat over R . □

This completes the proof of Theorem 1.1.

2 Comments on the proof

Rmk. 2.1. Several implications admit useful strengthenings.

- (i) For a finitely presented module, projectivity is local: in conditions (3) and (4), “free” may be replaced by “projective.” The proof using Lemma 1.4 needs only local projectivity.
- (ii) Over a local ring, every finitely generated flat module is free; one can weaken finite presentation in Lemma 1.3. A proof of this stronger statement uses the equational criterion for flatness. With no finiteness hypothesis, Kaplansky’s theorem says that every projective module over a local ring is free.

The proof is a compact meeting place for several techniques in commutative algebra. First, properties such as finite generation, finite presentation, and flatness behave well under extension of scalars; conversely, under faithful flatness they descend. Second, finite presentation has the stronger characterization used above: for every surjection $R^n \rightarrow M$, not merely for one chosen presentation, the kernel is finitely generated. Proving this carefully uses a comparison diagram and the snake lemma. Third, localization is transitive: for multiplicative subsets S, T ,

$$(A_S)_{T_S} \cong A_{ST}.$$

The implication $(2) \Rightarrow (3)$ is essentially Nakayama’s lemma applied over each local ring; the Tor sequence isolates the kernel of a minimal generating set. The implication $(3) \Rightarrow (1)$ is of a different flavor: it uses the finite-presentation hypothesis precisely to commute Hom with localization. Similar caution is needed elsewhere—Hom and tensor product do not commute in arbitrary circumstances. The same kernel-comparison technique appears, for example, when proving that completion commutes with tensor product under Noetherian hypotheses.

The steps involving condition (5) emphasize principal localizations. The key passage from a maximal ideal to a single function is Lemma 1.5. The reverse implication is especially useful conceptually: a ring is approximated by finitely many principal localizations in the faithfully flat sense, and module-theoretic properties can then be checked on that finite cover. The presentation here follows the corresponding discussion in Bosch [1, pp. 133–136].

References

- [1] S. Bosch, *Algebraic Geometry and Commutative Algebra*, Universitext, Springer, London, 2013, doi:10.1007/978-1-4471-4829-6.