

Several Theorems of Hua Lo-Keng

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This note continues our study of a less familiar group of exercises in Jacobson's *Basic Algebra I* [1]. The results are considerably better known than those in the earlier notes. Two principal statements—the Cartan–Brauer–Hua theorem and Hua's theorem on inverse-preserving maps—have especially elegant proofs due to Hua Lo-Keng. Both proofs rest on the two striking identities in Theorem 1.1.

1 Hua's identities and the Cartan–Brauer–Hua theorem

Thm. 1.1 (Hua's identities). Let R be a ring and let $a, b \in R$. Whenever all displayed inverses exist,

$$\left((a - b^{-1})^{-1} - a^{-1}\right)^{-1} = aba - a, \quad (1)$$

$$a = \left(b - (a - 1)^{-1}b(a - 1)\right)\left(a^{-1}ba - (a - 1)^{-1}b(a - 1)\right)^{-1}. \quad (2)$$

In (??) it is enough to require that a , b , and $ab - 1$ be invertible. In a division ring, the inverse in (1) exists when $ab \neq ba$.

Proof. For (??), verify that $aba - a$ is both a right and a left inverse:

$$\begin{aligned}
& ((a - b^{-1})^{-1} - a^{-1})(aba - a) \\
&= (a - b^{-1})^{-1}(ab - 1)a - (ba - 1) = ba - (ba - 1) = 1, \\
& (aba - a)((a - b^{-1})^{-1} - a^{-1}) \\
&= a(ba - 1)(a - b^{-1})^{-1} - (ab - 1) = ab - (ab - 1) = 1.
\end{aligned}$$

For (1), it suffices to check

$$\begin{aligned}
a(a^{-1}ba - (a - 1)^{-1}b(a - 1)) &= ba - (a - 1)(a - 1)^{-1}b(a - 1) \\
&\quad - (a - 1)^{-1}b(a - 1) \\
&= b - (a - 1)^{-1}b(a - 1).
\end{aligned}$$

□

Thm. 1.2 (Cartan–Brauer–Hua). Let D be a division ring with center C , and let S be a division subring invariant under every inner automorphism $x \mapsto dxd^{-1}$, $d \in D^\times$. Then either $S = D$ or $S \subseteq C$.

Hua's proof. Fix $b \in S$. By (1), every $a \in D$ for which $ab \neq ba$ is generated, using division-ring operations, by b and conjugates of b . Hence $a \in S$. Equivalently, every $a \notin S$ commutes with every element of S .

Suppose $S \not\subseteq C$. Choose $b \in S$ and $b' \in D$ with $bb' \neq b'b$. The preceding observation gives $b' \in S$. If $S \neq D$, choose $a \notin S$. Then $ab \notin S$, so both a and ab commute with b' . Thus

$$(ab)b' = b'(ab) = ab'b,$$

contrary to $bb' \neq b'b$. □

Brauer's proof [4]. Take $a \in S$ and $b \in D$. Invariance supplies $b_1, b_2 \in S$ such that

$$ba = ab_1, \quad b(1 + a) = (1 + a)b_2.$$

Subtracting gives $b - b_2 = a(b_2 - b_1)$. If $b_2 - b_1 \neq 0$, right multiplication by its inverse yields $a \in S$. Consequently, for every $a \notin S$ we must have $b_2 = b_1$, and then $b = b_2 = b_1$; in other words, every element outside S commutes with every element of D .

If $S \not\subseteq C$, choose $b \in S$ and $c \in D$ with $bc \neq cb$. The observation implies $c \in S$. If $S \neq D$, choose $a \notin S$. Both a and $a + c$ lie outside S , hence both commute with b , which contradicts $bc \neq cb$. $\square$

The second proof is included because it is closer to the ordinary range of algebraic experimentation than the sudden use of Hua's identity. Brauer's original note is [4].

2 Jordan homomorphisms

Def. 2.1. An *antihomomorphism* $\phi : R \rightarrow R'$ is a homomorphism of additive groups satisfying $\phi(1) = 1'$ and $\phi(ab) = \phi(b)\phi(a)$. A *Jordan homomorphism* $\eta : R \rightarrow R'$ is an additive map satisfying $\eta(1) = 1'$ and

$$\eta(aba) = \eta(a)\eta(b)\eta(a).$$

Lem. 2.2. Every Jordan homomorphism satisfies

- (1) $\eta(a^k) = \eta(a)^k$ for all $k \in \mathbb{N}$;
- (2) $\eta(abc + cba) = \eta(a)\eta(b)\eta(c) + \eta(c)\eta(b)\eta(a)$;
- (3) $\eta(ab + ba) = \eta(a)\eta(b) + \eta(b)\eta(a)$.

Proof. The first assertion is immediate for $k = 1, 2$ and then follows by induction from

$$\eta(a^n) = \eta(a a^{n-2} a) = \eta(a)\eta(a)^{n-2}\eta(a).$$

For the second, expand $\eta((a + c)b(a + c))$ in two ways. On one hand it equals

$$(\eta(a) + \eta(c))\eta(b)(\eta(a) + \eta(c));$$

on the other it equals $\eta(aba) + \eta(cbc) + \eta(abc + cba)$. Cancelling the first two terms proves (2).

Taking $c = 1$ proves (3). $\square$

Thm. 2.3 (Jacobson–Rickart). Let $\eta : R \rightarrow D$ be a Jordan homomorphism into a ring D without zero divisors; multiplication in D need not be commutative. For every $a, b \in R$, either

$$\eta(ab) = \eta(a)\eta(b) \quad \text{or} \quad \eta(ab) = \eta(b)\eta(a).$$

Proof. Using Lemma 2.2, calculate

$$\begin{aligned} & (\eta(ab) - \eta(a)\eta(b))(\eta(ab) - \eta(b)\eta(a)) \\ &= \eta(ab)^2 - \eta(a)\eta(b)\eta(ab) - \eta(ab)\eta(b)\eta(a) + \eta(a)\eta(b)^2\eta(a) \\ &= \eta(abab) - \eta(abab + abba) + \eta(abba) = 0. \end{aligned}$$

Since D has no zero divisors, one of the two factors is zero. □

3 Local multiplicativity becomes global

Thm. 3.1 (Hua). Suppose $\eta : R \rightarrow R'$ is additive, $\eta(1) = 1'$, and, for every $a, b \in R$, either

$$\eta(ab) = \eta(a)\eta(b) \quad \text{or} \quad \eta(ab) = \eta(b)\eta(a).$$

Then η is either a homomorphism or an antihomomorphism (and it may be both, for example when the image is commutative).

Proof. We first record two facts.

(i) If $\eta(xy) = \eta(x)\eta(y)$, then $\eta(yx) = \eta(y)\eta(x)$. Indeed, every such map satisfies $\eta(w^2) = \eta(w)^2$.

Expanding $\eta((x + y)^2)$ and comparing terms proves the assertion.

(ii) If

$$\eta(ab) = \eta(a)\eta(b) \neq \eta(b)\eta(a),$$

then $\eta(ac) = \eta(a)\eta(c)$ for every $c \in R$. Compare the two possible forms of $\eta(a(b + c))$ with $\eta(ab) + \eta(ac)$. If $\eta(ac)$ occurred only in the reversed order, either choice for $\eta(a(b + c))$ would contradict the displayed strict inequality.

Assume first that there is a noncommuting multiplicative pair a, x , so that

$$\eta(ax) = \eta(a)\eta(x) \neq \eta(x)\eta(a).$$

By (ii), $\eta(ay) = \eta(a)\eta(y)$ for every y . By (i), $\eta(xa) = \eta(x)\eta(a)$; applying (ii) with x in place of a shows that $\eta(xy) = \eta(x)\eta(y)$ for every y .

It remains to treat x with $\eta(x) \in C(\eta(a))$. If $\eta(x)$ is noncentral in the image, choose t such that $\eta(t)$ commutes with neither $\eta(a)$ nor $\eta(x)$. Such a t exists because the union of two proper additive subgroups cannot be the whole additive group. Applying the preceding argument first to a, t and then to x, t proves multiplicativity for x as well. Central image elements cause no distinction between the two orders. Thus η is multiplicative everywhere. If the initial noncommuting pair occurs in the reversed order, the same argument proves that η is antimultiplicative. If there is no noncommuting pair, both conclusions hold. $\square$

The proof above follows the local-to-global idea: first propagate the color of a noncommuting pair along every edge incident with one endpoint, then use an auxiliary element outside two centralizers to reach the remaining pairs. Jacobson and Rickart give a compact alternative [5]. One may picture the elements of R as vertices, coloring the edge $a-b$ red when multiplication is preserved and blue when it is reversed. A purely red edge forces all incident edges to be red or purple (both colors). If a purely blue edge remained, applying the rule to the factorization

$$ab + bc + ac + cd = (a + d)(b + c)$$

would produce a contradiction.

Thm. 3.2 (Jacobson–Rickart). A Jordan homomorphism from a ring R to a ring without zero divisors is either a homomorphism or an antihomomorphism.

Proof. Combine Theorems 2.3 and 3.1. $\square$

Thm. 3.3 (Hua). Let D, D' be division rings. Suppose $\eta : D \rightarrow D'$ is additive, $\eta(1) = 1'$, $\eta(a) \neq 0$ for $a \neq 0$, and

$$\eta(a^{-1}) = \eta(a)^{-1} \quad (a \neq 0).$$

Then η is either a homomorphism or an antihomomorphism.

Proof. Apply η to (??):

$$\eta(((a - b^{-1})^{-1} - a^{-1})^{-1}) = \eta(aba - a).$$

Additivity and preservation of inverses turn the left-hand side into

$$((\eta(a) - \eta(b)^{-1})^{-1} - \eta(a)^{-1})^{-1}.$$

Another application of (??), now in D' , shows that this is $\eta(a)\eta(b)\eta(a) - \eta(a)$. Hence

$$\eta(aba) = \eta(a)\eta(b)\eta(a),$$

so η is a Jordan homomorphism. The conclusion follows from Theorem 3.2. $\square$

4 Projective geometry and historical remarks

Projective geometry can be developed over an arbitrary division ring, just as real projective geometry is developed over $\mathbb{R}$. Harmonic ranges still make sense, and the fourth harmonic point is determined from the first three by

$$d = a - (2(a - b)^{-1} + (c - a)^{-1})^{-1}.$$

Together with Theorem 3.3, this leads to the following form of von Staudt's theorem; see Artin [6].

Thm. 4.1 (von Staudt). Let σ be a bijection of a projective line that sends every finite harmonic range to a finite harmonic range. In affine coordinates,

$$\sigma(x) = a\tau(x) + b,$$

where $a, b \in k, a \neq 0$, and τ is either an automorphism or an antiautomorphism of the underlying division ring k .

We close with some context for Hua's identities. The first identity is related to the familiar formula

$$(1 - ab)^{-1} = 1 + a(1 - ba)^{-1}b.$$

Indeed,

$$(a - aba)^{-1} = a^{-1} + (b^{-1}(1 - ba))^{-1} = a^{-1} - (a - b^{-1})^{-1}.$$

This explains how one might seek an expression for aba using inverses alone.

The attribution is not completely transparent. The identity (??) appears explicitly in Hua's 1949 paper [2]. Jacobson attributes (1) to Hua, while Artin's 1957 book [6] gives the identity while introducing Hua's theorem. Hua's paper on automorphisms [3] proves the division-ring version of Theorem 3.2 and relates it to projective geometry, but the present author did not locate there a printed proof of the exact formulation in Theorem 3.1.

One route to (1) begins with the multiplicative commutator $(a, b) = a^{-1}bab^{-1}$. The relation

$$a(a, b) = (a - 1)(a - 1, b) + 1$$

gives

$$a((a, b) - (a - 1, b)) = 1 - (a - 1, b),$$

and right multiplication by b yields the desired identity after rearrangement.

References

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