

A Double Cover of $\mathrm{SO}(4)$

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1 The quaternionic action

Problem 1.1. Let

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$$

be the quaternion division algebra, and let $\alpha, \beta \in \mathbb{S}^3 \subset \mathbb{H}$ be unit quaternions. Prove that

$$f_{\alpha, \beta} : \mathbb{H} \longrightarrow \mathbb{H}, \quad v \longmapsto \alpha v \bar{\beta}, \tag{1}$$

is an orientation-preserving isometry. Show that

$$\Phi : \mathbb{S}^3 \times \mathbb{S}^3 \longrightarrow \mathrm{SO}(4), \quad (\alpha, \beta) \longmapsto f_{\alpha, \beta}, \tag{2}$$

is a surjective homomorphism with kernel

$$\{(1, 1), (-1, -1)\}.$$

Thus (2) is a two-to-one covering homomorphism.

Proof. We identify a quaternion with its coordinate vector in $\mathbb{R}^4$. For $q \in \mathbb{H}$, write $|q|^2 = q\bar{q}$.

Multiplicativity of the norm gives

$$|\alpha v \bar{\beta}| = |\alpha| |v| |\beta| = |v|, \tag{3}$$

so (1) is an $\mathbb{R}$ -linear isometry.

For $\alpha = a + bi + cj + dk$, left multiplication by α has matrix

$$L_\alpha = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}. \quad (4)$$

A direct calculation gives

$$\det L_\alpha = (a^2 + b^2 + c^2 + d^2)^2 = 1. \quad (5)$$

Right multiplication by $\bar{\beta}$ has the same determinant. Hence $f_{\alpha,\beta} \in \text{SO}(4)$.

The homomorphism property follows from

$$\begin{aligned} f_{\alpha_1\alpha_2,\beta_1\beta_2}(v) &= (\alpha_1\alpha_2)v\overline{\beta_1\beta_2} \\ &= \alpha_1(\alpha_2v\bar{\beta}_2)\bar{\beta}_1 = f_{\alpha_1,\beta_1}(f_{\alpha_2,\beta_2}(v)). \end{aligned} \quad (6)$$

We next determine the kernel. If $\alpha v\bar{\beta} = v$ for every v , taking $v = 1$ gives $\alpha = \beta$. The remaining identity $\alpha v = v\alpha$ says that α lies in the center of $\mathbb{H}$, which is $\mathbb{R}$. Since $|\alpha| = 1$, we have $\alpha = \beta = 1$ or $\alpha = \beta = -1$. Thus

$$\ker \Phi = \{(1, 1), (-1, -1)\}. \quad (7)$$

It remains to prove surjectivity. We first isolate the familiar three-dimensional quaternionic rotation.

Lem. 1.2. The map

$$\rho : \mathbb{S}^3 \longrightarrow \text{SO}(\text{Im } \mathbb{H}) \cong \text{SO}(3), \quad q \longmapsto (x \mapsto qx\bar{q}),$$

is surjective.

Proof. Write $q = \cos(\theta/2) + u \sin(\theta/2)$ with $u \in \text{Im } \mathbb{H}$, $|u| = 1$. For $x \in \text{Im } \mathbb{H}$, use

$$ux = -u \cdot x + u \times x \quad (8)$$

to obtain Rodrigues' formula

$$qx\bar{q} = x \cos \theta + (u \times x) \sin \theta + u(u \cdot x)(1 - \cos \theta). \quad (9)$$

This is rotation through angle θ about the axis u . Every element of $\text{SO}(3)$ is such a rotation. $\square$

Now take $T \in \text{SO}(4)$ and set $v_1 = T(1) \in \mathbb{S}^3$. Left multiplication by $\bar{v}_1$ is in $\text{SO}(4)$, so

$$T' = L_{\bar{v}_1} \circ T \in \text{SO}(4), \quad T'(1) = 1.$$

An orthogonal transformation fixing 1 preserves its orthogonal complement $1^\perp = \text{Im } \mathbb{H}$. The restriction $T'|_{\text{Im } \mathbb{H}}$ is in $\text{SO}(3)$. By Lemma 1.2, choose $q \in \mathbb{S}^3$ such that

$$T'(x) = qx\bar{q} \quad (x \in \text{Im } \mathbb{H}).$$

The same formula holds for $x = 1$, hence for every quaternion by linearity. Therefore

$$T(v) = v_1qv\bar{q} = f_{v_1q,q}(v),$$

which proves that (2) is surjective.

The domain and codomain are compact smooth Lie groups of the same dimension, and the kernel (7) is discrete. Thus Φ is a local diffeomorphism and a two-sheeted covering map. $\square$

2 The coordinate calculation behind surjectivity

The preceding proof packages the longest part of a direct calculation into Lemma 1.2. For completeness, we record the coordinate route that motivated the argument.

Let $e_1 = 1, e_2 = i, e_3 = j, e_4 = k$, and put $v_r = T(e_r)$. To solve

$$T(v) = \alpha v \bar{\beta}$$

it is enough to solve the four equations

$$\alpha = v_1\beta, \quad v_1\beta i = v_2\beta, \quad v_1\beta j = v_3\beta, \quad v_1\beta k = v_4\beta. \quad (10)$$

After multiplying by $\bar{v}_1$, define

$$z_r = \bar{v}_1 v_r \quad (r = 2, 3, 4). \quad (11)$$

Then the last three equations in (10) become

$$\beta i \bar{\beta} = z_2, \quad \beta j \bar{\beta} = z_3, \quad \beta k \bar{\beta} = z_4. \quad (12)$$

For arbitrary quaternions

$$x = a_1 + a_2i + a_3j + a_4k, \quad y = b_1 + b_2i + b_3j + b_4k,$$

one computes

$$\begin{aligned} \bar{x}y &= x \cdot y + (a_1b_2 - a_2b_1 - a_3b_4 + a_4b_3)i \\ &\quad + (a_1b_3 - a_3b_1 + a_2b_4 - a_4b_2)j \\ &\quad + (a_1b_4 - a_4b_1 - a_2b_3 + a_3b_2)k. \end{aligned} \quad (13)$$

Since $v_1 \perp v_r$, Equation (13) shows that z_r is purely imaginary. Lagrange's identity gives

$$\sum_{1 \leq r < s \leq 4} (a_rb_s - a_sb_r)^2 = |x|^2|y|^2 - (x \cdot y)^2, \quad (14)$$

so $|z_r| = 1$. Hence $z_r \in \mathbb{S}^2 \subset \text{Im } \mathbb{H}$ and $z_r^2 = -1$.

The three vectors are orthonormal. For instance, using

$$x \cdot y = -\frac{1}{2}(xy + yx) \quad (x, y \in \text{Im } \mathbb{H}), \quad (15)$$

we obtain

$$\begin{aligned} z_2 \cdot z_3 &= -\frac{1}{2}(\bar{v}_1v_2\bar{v}_1v_3 + \bar{v}_1v_3\bar{v}_1v_2) = 0, \\ z_2 \cdot z_2 &= -\bar{v}_1v_2\bar{v}_1v_2 = \bar{v}_2v_1\bar{v}_1v_2 = 1. \end{aligned}$$

The other cases are identical. Orientation can be checked by the scalar triple product:

$$\det(z_2, z_3, z_4) = (z_2 \times z_3) \cdot z_4 = \bar{v}_1v_2\bar{v}_3v_4 = 1. \quad (16)$$

The last equality is the quaternionic form of the complementary-minor identity

$$\det A_{I,J} = \det(A) \det((A^{-1})_{J^c, I^c}). \quad (17)$$

For $A \in \text{SO}(4)$, $A^{-1} = A^T$ and $\det A = 1$, so every 2×2 minor equals the appropriately signed complementary minor. Expanding (16) in minors gives 1.

Thus the matrix $Z = (z_2, z_3, z_4)$ belongs to $\text{SO}(3)$. By Lemma 1.2, there is a unit quaternion β satisfying (12); then $\alpha = v_1\beta$ solves (10). This completes the direct calculation without expanding the original problem into twelve scalar equations.

3 Spin groups

Rmk. 3.1. The coordinate proof is deliberately explicit, but the structural explanation is shorter. The Cartan–Dieudonné theorem says that every orthogonal transformation of an n -dimensional quadratic space is a product of at most n reflections. In quaternionic notation, reflection in the hyperplane perpendicular to a unit quaternion v can be written

$$x \mapsto -v\bar{x}v. \tag{18}$$

Composing an even number of such reflections gives the left–right action (1).

This is the four-dimensional instance of the spin construction:

$$\mathrm{Spin}(4) \cong \mathbb{S}^3 \times \mathbb{S}^3 \cong \mathrm{SU}(2) \times \mathrm{SU}(2), \tag{19}$$

and the canonical map $\mathrm{Spin}(4) \rightarrow \mathrm{SO}(4)$ is the double cover in Problem 1.1. More generally, the Clifford algebra of a quadratic space (V, Q) produces a two-to-one covering $\mathrm{Spin}(V, Q) \rightarrow \mathrm{SO}(V, Q)$ in the appropriate real signature.