

A Survey on 5 Decompositions Based on Relative Hard

Lefschetz

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0 Introduction

In this survey I shall go over the paper [1] of de Cataldo, where he summarizes 5 “canonical” decompositions of a filtered object in an abstract setting, in the filtered category $\mathcal{AF}$ associated with a given abelian category $\mathcal{A}$. The motivation for such a decomposition is clear from our discussion throughout the semester. We are considering a proper map $f : X \rightarrow Y$ between complex varieties and asking that whether the derived direct image $f_* IC_X[d]$ splits into direct sum of shifted perverse cohomology sheaves in the derived category $\mathcal{D}_Y$. After taking cohomology, we may ask the cohomological version of this question, but in the abelian category $\mathcal{V}_{\mathbb{Q}}$, consisting of finite dimensional rational vector spaces, that

$$H := \bigoplus_{d \geq 0} IH^d(X, \mathbb{Q})$$

splits or not. We know that f equips H with the perverse Leray filtration P , which is indeed canonical, and we can consider the corresponding graded objects $H_p := P_p H / P_{p-1} H$, being nontrivial only in the interval $[-r, r]$, where $r = r(f)$ denotes the so-called “defect of semismallness” of the map f . Thus, if we consider the corresponding filtered category $\mathcal{V}_{\mathbb{Q}} \mathcal{F}$, we get two objects

$$\mathbb{H} := (H, P), \quad \mathbb{H}_* = \bigoplus_p (H_p, T[-p]),$$

where for the exact definition, one may refer to the following section. To determine whether $\mathbb{H}$ splits is to ask whether there exists an isomorphism $\varphi : \mathbb{H} \rightarrow \mathbb{H}_*$ and further we may want this isomorphism to induce identity on each graded piece H_p , which will be called *good* later. What’s more, considering the natural mixed Hodge structure underlying H , one may want to ask if the same story happens for the mixed Hodge structure, i.e., in the abelian category $\mathcal{MHS}$ of mixed Hodge structures and the corresponding filtered category $\mathcal{MHS} \mathcal{F}$, whether there is a splitting $\varphi : \mathbb{H} \rightarrow \mathbb{H}_*$ such that the induced map on each component is identity. In view of such generalities, things are to be done abstractly in an arbitrary abelian category $\mathcal{A}$. The key input is the relative hard Lefschetz

theorem (abbreviated to be RHL hereafter). Consider an f -relative ample divisor η on X . The cup product with η induces an arrow $\mathbb{H} \rightarrow \mathbb{H}[2](1)$ in $\mathcal{MHST}$, namely, $\eta : H \rightarrow H(1)$ is such that $\eta : P_p H \rightarrow P_{p+2} H(1)$, where $(\cdot)$ means the Tate shift, to be explained in the following section. Then the usual version of hard Lefschetz implies an isomorphism of the cohomology vector spaces, but in fact we have more. RHL ensures isomorphisms in $\mathcal{MHST}$ that $\eta^k : H_{-k} \cong H_k(K)$ for every $k \geq 0$. The point of paper [1] and this survey is that, in any abstract abelian category $\mathcal{A}$, as long as we have knowledge of such isomorphisms given by “RHL”, one can construct good splittings $\varphi : \mathbb{H} \rightarrow \mathbb{H}_*$ in $\mathcal{AF}$. For an “RHL” in our abstract setting now, we mean a pair $(\mathbb{V}, e)$ where $e : \mathbb{V} \rightarrow \mathbb{V}[2](1)$ is a morphism in $\mathcal{AF}$ such that it induces isomorphisms $e^k : V_{-k} \rightarrow V_k(k)$.

The construction starts from a most elementary one ω_I , which is discovered in paper [1], and recover the other 4 splittings in knowledge of such a splitting ω_I .

1 Basic definitions and notations

It is inevitable to introduce a heavy load of definitions and concepts to start the discussion, which is to be done in this section. This section is almost a duplication of the corresponding section in paper [1], except for some personal reminders and caveats that have been made to make sure the later explicit calculations would not mismatch.

1.1 Filtered category and graded objects

Let $\mathcal{A}$ be an Abelian category whose elements we will denote by V, W , etc. A filtration F on V is a finite increasing filtration

$$\dots \subseteq F_p V \subseteq F_{p+1} V \subseteq \dots \subseteq$$

of V such that $F_p V = 0$ for $p \ll 0$ and $F_p V = V$ for $p \gg 0$. Denote $\text{Gr}_p^F V := F_p V / F_{p-1} V$ and we denote by T the trivial filtration on V : $T_{-1} V = 0 \subseteq T_0 V = V$. Given $n \in \mathbb{Z}$, we denote by $F[n]$ the n -th translate of F : $F[n]_p V := F_{n+p} V$, so $\text{Gr}_p^{F[n]} V = \text{Gr}_{n+p}^F V$. For example $T[-p]$ is the trivial filtration in position p . Given a pair $\mathbb{V} := (V, F)$ and a subquotient U of V , the filtration F on V induces a filtration on U , which we still denote by F . For example, when we write $(\text{Gr}_p^F V, F)$, we really means that to be $(\text{Gr}_p^F V, T[-p])$. In particular, for every $p \leq q \in \mathbb{Z}$, we have the following pairs associated with $\mathbb{V}$:

$$\begin{aligned} \mathbb{V}_{\leq p} &:= (F_p V, F), & \mathbb{V}_{\geq p} &:= (V / F_{p-1} V, F), \\ \mathbb{V}_p &:= (\text{Gr}_p^F V, F), & \mathbb{V}_{[p,q]} &:= (F_q V / F_{p-1} V, F). \end{aligned}$$

We say that $\mathbb{V}$ has *type* $[a, b]$, with $a \leq b$, if $\text{Gr}_p^F V = 0$ for every $p \notin [a, b]$. Given (V, F) and (W, F) , an arrow $\mathbb{I} : V \rightarrow W$ in $\mathcal{A}$ is said to be a *filtered arrow* if it respects the given filtrations, i.e., $\mathbb{I}$ maps $F_p V$ to $F_p W$, for every p . The filtered category $\mathcal{AF}$ associated with $\mathcal{A}$ is the category with objects the pairs $\mathbb{V} = (V, F)$ and arrows the filtered arrows. In particular, an arrow $\mathbb{I} : \mathbb{V} \rightarrow \mathbb{W}$ induces arrows on the objects listed above, for example,

$l_i : \mathbb{V}_i \rightarrow \mathbb{W}_i$. An arrow l in $\mathcal{AF}$ is an *isomorphism* if and only if l_i is an isomorphism for every $i \in \mathbb{Z}$. Now as usual we have shift functors $[n] : \mathcal{AF} \rightarrow \mathcal{AF}$ such that $(V, F) = \mathbb{V} \mapsto \mathbb{V}[n] := (V, F[n])$, etc.

Now consider the following graded objects, in $\mathcal{A}$, $\mathcal{AF}$ and $\mathcal{AF}$, respectively:

$$V_* := \bigoplus_p V_p, \quad \mathbb{V}_* := \bigoplus_p \mathbb{V}_p.$$

We say that $\mathbb{V}$ *splits* in $\mathcal{AF}$ if there is an isomorphism in $\mathcal{AF}$: $\varphi : \mathbb{V}_* \xrightarrow{\cong} \mathbb{V}$, and we say that a splitting φ is *good* if for every p , the induced map $\varphi_p : V_p \cong V_p$ is the identity.

1.2 Tate shift

What differs from the case when we are only considering splitting using relative hard Lefschetz in the case of cohomology vector spaces is that, when discussing about mixed Hodge structures and their splitting, we must take into consideration the weights. This lead to the Tate shift functor, which is denoted by (1) . It means that if $\mathbb{Z}$ is the pure Hodge structure with weight zero and type $(0, 0)$, then $\mathbb{Z}(1)$ is the pure Hodge structure with weight -2 and type $(-1, -1)$. For example, the cup product with the first Chern class L of a line bundle on a complex algebraic variety X induces, for every $k \geq 0$, a map $L : IH^k(X, \mathbb{Z}) \rightarrow IH^{k+2}(X, \mathbb{Z})(1)$. In general, let $(1) : \mathcal{A} \rightarrow \mathcal{A}$, $V \mapsto V(1)$, $l \mapsto l(1)$, be an additive and exact autoequivalence and, for $m \in \mathbb{Z}$, denote by (m) its m -th iterate, called the m -shift functor. By exactness, the shift functors lift to functors (also called shift functors):

$$(m) : \mathcal{AF} \rightarrow \mathcal{AF}, \quad \mathbb{V} = (V, F) \mapsto (V(m), F) =: \mathbb{V}(m), \quad l \mapsto l(m),$$

and we have

$$(\mathbb{V}(m))_p = \mathbb{V}_p(m), \quad \text{Gr}_p^F(l(m)) = (\text{Gr}_p^F l)(m).$$

The shift functors commute with the translation functors, so that we can write $\mathbb{V}[n](m)$ unambiguously. We have $(\mathbb{V}[n](m))_p = \mathbb{V}_{n+p}(m)$, etc.

1.3 Arrows and their degrees

For every $p \in \mathbb{Z}$, an arrow $l : V \rightarrow W[n](m)$ induces arrows of graded parts in $\mathcal{A}$:

$$l_p : V_p \longrightarrow W_{n+p}(m),$$

where $W_{n+p}(m) = \text{Gr}_{n+p}^F(m) = \text{Gr}_p^{F[n]}W(m)$. If $\mathbb{V}$ has type $[a, b]$, then we have canonical arrows:

$$\mathbb{V}_a \xrightarrow{i_a} \mathbb{V} \xrightarrow{p_b} \mathbb{V}_b,$$

first inclusion and last quotient, with composition $\partial_{ab}\text{Id}$. This observation will be used in the elementary construction of ω_l . An arrow $l[n](m)$ obtained from an arrow $l : \mathbb{V} \rightarrow \mathbb{W}$ by shift/translation is simply denoted by l .

(For example, when we are writing the arrow $\mathbb{I}_a^{-1}$, it really means $\mathbb{I}_a^{-1}(-m)$. This notation will turn up later in the construction of ω_I .)

Now we come to the filtered category. Let $\mathbb{V}$ and $\mathbb{W}$ be in $\mathcal{AF}$, let $m, n \in \mathbb{Z}$, let $\mathbb{I} : \mathbb{V}_* \rightarrow \mathbb{W}_*[n](m)$ be an arrow in $\mathcal{AF}$ and let $\mathbb{I}_{pq} : \mathbb{V}_p \rightarrow \mathbb{W}_q[n](m)$ be the (p, q) -th component of $\mathbb{I}$. We define the degree $d \in \mathbb{Z}$ homogeneous part $\mathbb{I}^{\{d\}}$ of $\mathbb{I}$ by setting:

$$\mathbb{I}^{\{d\}} := \sum_{q-p=d} \mathbb{I}_{pq}, \quad \left(\mathbb{I} = \sum_d \mathbb{I}^{\{d\}} \right).$$

Since $\mathbb{V}_p = (V_p, T[-p])$ and $\mathbb{W}_q[n](m) = (W_q(m), T[-q+n])$, basically we find that

$$\mathbb{I}^{\{d\}} = 0, \quad \forall d \geq n+1.$$

1.4 Dual categories

We need to introduce the dual category of an abelian variety and of its corresponding filtered variety for the construction of splittings ϕ_{II} and ω_{II} , which are dual to ϕ_I and ω_I , respectively. Let $\mathcal{A}^o$ denote the Abelian category opposite to $\mathcal{A}$. (We will use in section 3, in concrete calculations of vector spaces, i.e., when $\mathcal{A} = \mathcal{V}_{\mathbb{Q}}$ is the category of finite dimensional rational vector spaces and linear maps, that the natural contravariant functor $\mathcal{V}_{\mathbb{Q}} \rightarrow \mathcal{V}_{\mathbb{Q}}^o$ can be identified with taking dual vector spaces and transposition of linear maps.)

We have the exact anti-equivalence $(-)^o : \mathcal{A} \rightarrow \mathcal{A}^o$, $(V \xrightarrow{f} W) \mapsto (V^o \xleftarrow{f^o} W^o)$ which is an involution by definition. We can equip $\mathcal{A}^o$ with the additive and exact shift functors $(m)^o : V^o \mapsto V^o(m)^o := (V(-m))^o$ naturally. Now the dual things carry out to the filter category directly and naturally. For later use, we write down the explicit definitions of some ingredients to be used later. A filtered object $\mathbb{V} = (V, F)$ in $\mathcal{AF}$ gives rise to a filtered object $\mathbb{V}^o = (V^o, F^o)$ in $\mathcal{A}^o\mathcal{F}$ by setting $F_i^o V^o = (V^o)_{\leq i} := (V_{\geq -i})^o$.

Hence we obtain an anti-equivalence $(-)^o : \mathcal{AF} \rightarrow \mathcal{A}^o\mathcal{F}$ which is an involution. The anti-equivalence $(-)^o$ is anti-compatible with translations, shifts and taking graded pieces, etc., for example: $\mathbb{V}^o[n]^o(m)^o = (\mathbb{V}[-n](-m))^o$. An arrow $\mathbb{I} : \mathbb{V} \rightarrow \mathbb{W}[n](m)$ in $\mathcal{AF}$ yields the arrow $\mathbb{I}^o : \mathbb{V}^o \rightarrow \mathbb{W}^o[n]^o(m)^o$ in $\mathcal{A}^o\mathcal{F}$ which really means $\mathbb{I}^o[n]^o(m)^o$, but in notations later, we will omit these shifts as in the previous section.



Note that the dual of an arrow of degree d still has degree d instead of $-d$ as seen in the previous paragraph! This will lead to confusion for the condition later we will impose on ϕ_{III} if not think prudently. To avoid mistake, one can think of this more directly by considering that a dual arrow sends an element of degree $-n$ to degree 0, if the original arrow sends from 0 to n , hence with the same degree n .

1.5 Matrix representations and primitive Lefschetz decomposition

Finally we shall introduce the matrix representation of an arrow with respect to a splitting, which will be used in our later computations of concrete examples. These also provide a more down-to-earth way to look at the

characterizing properties of the different splittings constructed. And indeed some characterizations will be stated in the language of matrices in later sections. Let $\varphi : \mathbb{V}_* \cong \mathbb{V}$ be a splitting. The matrix $\tilde{\mathbf{I}}(\varphi)$ of $\mathbf{I}$ (with the same assumption as in the previous section) with respect to φ is defined by setting:

$$\tilde{\mathbf{I}}(\varphi) = \tilde{\mathbf{I}} := (\varphi^{-1} \circ \mathbf{I} \circ \varphi) : \mathbb{V}_* \longrightarrow \mathbb{V}_*[2](1), \quad \tilde{\mathbf{I}} = \sum_{pq} \tilde{\mathbf{I}}_{pq} = \sum \tilde{\mathbf{I}}^{\{d\}}.$$

The basic observation before that $\mathbf{I}^{\{d\}} = 0$, for all $d \geq n + 1$ translates into that $\tilde{\mathbf{I}}^{\{d\}} = 0$, *forall* $d \geq n + 1$.

In practice, this arrow $\mathbf{I}$ will be taken to be e , which is to be introduced in the next section.

2 Construction of the 5 decompositions in general

In the following sections, let $\mathbb{V} = (V, F)$ in $\mathcal{AF}$ be of type $[-r, r]$ for some $r \geq 0$. This can always hold true if we translate the index properly in our originally concerned examples and in what follows. Let

$$e : \mathbb{V} \longrightarrow \mathbb{V}[2](1)$$

be an arrow in $\mathcal{AF}$. In particular, for every $k \geq 0$, we have the iterations e^k and their graded parts:

$$e^k : \mathbb{V} \longrightarrow \mathbb{V}[2k](k), \quad e^k = V_j \longrightarrow V_{j+2k}(k).$$



One should notice that, though denoted both by e , the original operator $\mathbb{V} \rightarrow \mathbb{V}[2](1)$ may be different from the graded part $V_i \rightarrow V_{i+2}$. Using the language of the d -th degree part as in the previous section 1.5, the latter is in fact the “pure” degree 2 part of the original operator. We would see in later sections that this non-purity of degree will lead to almost all the troubles and non-duality in the constructions, also leading us to a way to find the self-dual splitting ϕ_{III} .

As is pointed out in the introduction, the key assumption is that $(\mathbb{V}, e)$ satisfies the following hard Lefschetz-type condition (HL):

Assumption 2.3.1 (Condition HL) The arrows:

$$e^k : V_{-k} \xrightarrow{\cong} V_k(k), \quad \forall k \geq 0,$$

are isomorphisms in $\mathcal{A}$.

2.1 de Cataldo’s decompositions ω_I and ω_{II}

First we construct a decomposition ω_I and its dual ω_{II} . This is based on a very elementary linear algebraic observation, that is, we can construct a complement subspace via the kernel of a left inverse. Strictly speaking, we have the following lemma:

Lem 2.1. Let $\mathcal{B}$ be an additive category and $\rho : B \rightarrow B'$ be an arrow in $\mathcal{B}$. Suppose $\ker \rho$ exists, i.e., there is a monomorphism $\iota_\rho : \ker \rho \rightarrow B$, and there exists a right inverse $r : B' \rightarrow B$ of ρ such that $\rho \circ r = \text{id}_{B'}$, then the natural arrow

$$B' \oplus \ker \rho \xrightarrow{r + \iota_\rho} B$$

is an isomorphism in $\mathcal{B}$.

Proof. Now since $\rho \circ (1 - r\rho) = 0$, there is an unique arrow $s : B \rightarrow \ker \rho$ such that $1 - r\rho = \iota_\rho \circ s$. And one can check that (ρ, s) gives a desired inverse to $r + \iota_\rho$. $\square$



Note that in the statement of the proof, we do not assume the existence of a kernel is automatic since for our later application of the lemma in $\mathcal{AF}$, which is not abelian in general, we need to impose this extra requirement.

Now using the HL condition of e , which gives an isomorphism from V_{-k} to V_k , we can try to “peel off” these two vector subspaces simultaneously by constructing complement to them, using the previous lemma and carry on by induction. Now we try to calculate the explicit left inverse. This will be done in a more general setting: suppose $\mathbb{V} = (V, F) \in \mathcal{AF}$ is of type $[a, b]$, $m \in \mathbb{Z}$, and $\mathbb{I} : \mathbb{V} \rightarrow \mathbb{V}[b-a](m)$ is an arrow inducing an isomorphism on graded parts: $\mathbb{I}_a : V_a \xrightarrow{\sim} V_b(m)$. Denote by i_a the inclusion $\mathbb{V}_a \hookrightarrow \mathbb{V}$ (Of course for our use this $\mathbb{I}$ would just be taken to be e .)



Note that by this definition we can really distinguish $\mathbb{I}$ and its graded part $\mathbb{I}_a$ acting from V_a to V_b , in contrast with the previous mixed notation e . Also, by definition we have that $p_b \mathbb{I}_a = \mathbb{I}_a$, which is a relation that would be needed to verify the construction below.

Now the inclusion $r : \mathbb{V}_a \oplus \mathbb{V}_b \rightarrow \mathbb{V}$ is given by $i_a + \mathbb{I}_a \mathbb{I}_a^{-1}$. Recalling a very naive relation $p_b i_a = \delta_{ab} \text{Id}$ as observed before, we can use this to construct a desired left inverse ρ . Now we can try a very straightforward construction by $(\mathbb{I}_a^{-1} p_b \mathbb{I}, p_b)$, since this is indeed an isomorphism on the second summand $\mathbb{V}_b$ trivially by the relations $p_b i_a = 0$ and $p_b \mathbb{I}_a \mathbb{I}_a^{-1} = \text{id}$ as said before. However, on the first summand $\mathbb{V}_a$, one can check that there is one (and merely this one) extra error given by $\mathbb{I}_a^{-1} p_b \mathbb{I}^2 i_a \mathbb{I}_a^{-1}$. This is indeed a good news since we can directly fix this by a “Gram-Schmidt” process, namely, an upper triangular matrix. Summing up, the following (seemingly complicated by indeed simple) arrow

$$\rho : \mathbb{V} \rightarrow \mathbb{V}_a \oplus \mathbb{V}_b, x \mapsto (\mathbb{I}_a^{-1} p_b \mathbb{I} - \mathbb{I}_a^{-1} p_b \mathbb{I}^2 i_a \mathbb{I}_a^{-1}, p_b)x$$

yields the desired left inverse, and hence its kernel (denoted by $\mathbb{Ker} \rho$ since it lives in the filtered category) gives the complement:

Lem 2.2. The arrow $w = i_a + \iota_\rho + \mathbb{I}_a \mathbb{I}_a^{-1}$ is an isomorphism in $\mathcal{AF}$, where ρ is as above:

$$w : \mathbb{V}_a \oplus \mathbb{Ker} \rho \oplus \mathbb{V}_b \rightarrow \mathbb{V}.$$

Apply this to $e : \mathbb{V} \rightarrow \mathbb{V}[2](1)$ yields the first splitting $\omega_I(e)$. Carry out this process in the dual category for e° will in a same way imply a splitting $\omega_I(e^\circ) : \mathbb{V}_*^\circ \cong \mathbb{V}^\circ$ in the dual category. Dualizing it back gives an “anti-splitting”

in the original category since it is an isomorphism $\mathbb{V} \xrightarrow{\sim} \mathbb{V}_*$, hence we need to take another inverse to get a true splitting in the original category, which is the second splitting $\omega_{II}(e)$. In symbols, this is $((\omega_I(e^\circ))^\circ)^{-1}$, which may seem confusing so I have explained the construction in a little more detail here. For explicit calculation and work in the dual category, one may refer to example 3.1.

2.2 Deligne's decompositions ϕ_I and ϕ_{II}

Now we come to the first two splittings constructed by Deligne. They rely on the primitive Lefschetz decomposition given by the relative hard Lefschetz condition. In usual texts for Hodge theory, this may be proved together with the hard Lefschetz using the full strength of the $\mathfrak{sl}_2$ -triple. However, one can prove rather elementarily that as long as we have the hard Lefschetz (HL) condition, we have the primitive Lefschetz decomposition. We summarize this in a lemma.

Lem 2.3. Suppose we have a series of vectors spaces together with a degree 2 operator acting on them:

$$V_{-n} \xrightarrow{e} V_{-n+2} \xrightarrow{e} \dots \xrightarrow{e} V_{n-2} \xrightarrow{e} V_n,$$

and that $e^k : V_{-k} \rightarrow V_k$ is an isomorphism for every $k \geq 0$. Denote by P_{-l} the primitive part in V_{-l} , i.e., $\ker e^{l+1} \cap V_l$. Then for every l , we have an isomorphism $V_{-l} \cong P_{-l} \oplus e(V_{-l-2})$, and by induction we get an isomorphism (the primitive Lefschetz decomposition)

$$\bigoplus_{j \geq 0} e^j(P_{-i-2j}) \xrightarrow{\sim} V_{-i}, i \geq 0;$$

acting by e^i , which is an isomorphism from V_{-i} to V_i , we get the corresponding primitive Lefschetz decomposition for V_i , i.e.,

$$\bigoplus_{j \geq 0} e^{i+j}(P_{-i-2j}) \xrightarrow{\sim} V_i, i \geq 0.$$

Proof. We only need to prove the first statement and the remaining follows. We prove this by checking the intersection is 0 and counting the dimension. Now suppose $x \in P_{-l} \cap e(V_{-l-2})$, then $e^{l+1}x = e^{l+2}y = 0$ for $l \in V_{-l-2}$. However since e^{l+2} is an isomorphism from V_{-l-2} , one immediately finds that $x = y = 0$.

On the other hand, the series of HL conditions imply in particular that $e : V_{-l-2} \rightarrow V_{-l}$ is injective and $e : V_l \rightarrow V_{l+2}$ is surjective for $l \geq 0$. Then one finds that $\dim e(V_{-l-2}) = \dim V_{-l-2} = \dim V_{l+2}$ by HL condition again. And by the rank-nullity theorem, $\dim P_{-l} = \dim V_{-l} - \dim \text{im } e^{l+1}$. However, since e^l is an isomorphism from e^l and e is surjective onto V_{l+2} , we have $\text{im } e^{l+1} = V_{l+2}$. Together these implies that $\dim P_{-l} + \dim e(V_{-l-2}) = \dim V_{-l}$ and we are done. $\square$

Now we come to our situation. The details, especially the algebraic details of the characterizing properties of the decompositions will not be fully spoken out in this survey due to length limit, and also because I think one

may see what happens in the procedure by looking at example 3.1. In particular, the proof of all the lemmas and propositions will be omitted.

Let $(\mathbb{V}, e)$ assume that it satisfies the HL condition as before. Let $i \geq 0$ and define the *primitive objects* in $\mathcal{A}$:

$$P_{-i} := \text{Ker} \{e^{i+1} : V_{-i} \longrightarrow V_{i+2}(i+1)\}.$$

The subquotient P_{-i} of V inherits the filtration induced by F on V , i.e., the trivial filtration translated in position $-i$, and we denote the resulting object in $\mathcal{AF}$ by: $\mathbb{P}_{-i} = (P_{-i}, F) = (P_{-i}, T[i])$.



In view of the pervious lemma, we obtain the canonical primitive Lefschetz decompositions (PLD) in $\mathcal{A}$ and $\mathcal{AF}$, respectively, where we identify the image $e^j(P_{-i-2j})$ with P_{-i-2j} itself, and we have included the Tate shifts in the notation, hopefully not leading to trouble for notation:

$$\bigoplus_{0 \leq j \leq i} P_{-i}(-j) \xrightarrow[\cong]{e} V_*,$$

$$\bigoplus_{0 \leq j \leq i} (P_{-i}(-j), T[i-2j]) = \bigoplus_{0 \leq j \leq i} \mathbb{P}_{-i}(-j)[-2j] \xrightarrow[\cong]{e} \mathbb{V}_*.$$

Visually, one can think of the following commutative diagram of epimorphisms and monomorphisms in $\mathcal{AF}$:

$$\begin{array}{ccccc} & & \mathbb{P}_{-i} & \xrightarrow{\text{mono}} & \mathbb{V}_{-i} \\ & & \downarrow \text{mono} & \nearrow \text{mono} & \\ \mathbb{V} & \xrightarrow{\text{epi}} & \dots & \xrightarrow{\text{epi}} & \mathbb{V}_{\geq -i} \end{array}$$

In view of (or in use of) such primitive Lefschetz decomposition at hand, to construct a splitting $\mathbb{V}_* \xrightarrow{\sim} \mathbb{V}$, one only need to lift the elements in the primitive parts appropriately to $\mathbb{V}$, since one can first transit to the primitive parts by the isomorphism e^{-1} as constructed above. This is done in the following lemma.

Lem 2.4. Let $(\mathbb{V}, e)$ be as before. There is a unique arrow $f_i : \mathbb{P}_{-i} \rightarrow \mathbb{V}$ in $\mathcal{AF}$ lifting the natural arrow $\mathbb{P}_{-i} \rightarrow \mathbb{V}_{\geq -i}$ as in the previous diagram, such that for every $s > i \geq 0$, the composition of the arrows below is zero:

$$\mathbb{P}_{-i} \xrightarrow{e^s f_i} \mathbb{V}(s) \rightarrow \mathbb{V}_{\geq s}(s).$$

Formally, we define the *first Deligne isomorphism* $\phi_I(e)$ associated with $(\mathbb{V}, e)$ by taking the composition of the following two isomorphisms:

$$\phi_I(e) : \mathbb{V}_* \xrightarrow{e^{-1}} \bigoplus_{0 \leq j \leq i} \mathbb{P}_{-i}(-j)[-2j] \xrightarrow{\sum e^j f_i} \mathbb{V}.$$

Rmk 2.1. I would omit the proof of the lemma, only trying to explain how one can correctly think of the lift. For example, if we consider $i = 0$, the requirement is the strongest: one must ensure $e(f_i(v))$ lands exactly in $\mathbb{V}_0$ for $v \in \mathbb{P}_0$. However on the other hand, for $i = -r$ or even $-r - 2$ where $\mathbb{V}$ is of type $[-r, r]$, we may find the condition is empty in these cases! This would be useful in practice when calculating the examples in 3.1, since in these cases we always have a naive splitting giving the liftings from the lowest primitive parts. Then we need no

further regulation for them. Also, one may find immediately by this, when restricted to $\mathbb{V}_{-r} \oplus \mathbb{V}_r$, the splittings ω_I and ϕ_I coincide.

So one can intuitively think of f_i as an appropriate lifting of the primitive elements into $\mathbb{V}$. Now considering the primitive Lefschetz decomposition as strings starting from the primitive elements and going to higher degrees by action of e , one can similarly imagine that the first Deligne's decomposition is also given by acting e recursively, but on those “appropriately” lifted elements in $\mathbb{V}$, rather than the primitive elements themselves, also taking them from the bottom in a string in the primitive Lefschetz decomposition to the top. This would intuitively lead to the next lemma characterizing the splitting by its “triangular” property in matrix terms.

Now let's recall the matrix representation appeared in section 1.5 and as before, we are considering $\mathbb{I} = e$ now. First assume that φ is good, then

$$\tilde{e}^{\{2\}} = \sum_p e, \quad e : \mathbb{V}_p \xrightarrow{e} \mathbb{V}_{p+2}(1).$$

One may notice that while $\tilde{e}^{\{2\}}$ is independent of φ , we have that $\tilde{e}(\varphi)^{\{d\}}$ depends on φ for $d \leq 1$. We will calculate some concrete examples in section 3 to show how these low degree homogeneous terms behave in reality. Now we can also refine this taking the primitive Lefschetz decomposition into account. We define

$$\tilde{e}(\varphi) = \tilde{e} := (\epsilon^{-1} \circ \tilde{e} \circ \epsilon) : \bigoplus_{0 \leq j \leq i} \mathbb{P}_{-i}(-j)[-2j] \longrightarrow \bigoplus_{0 \leq j \leq i} \mathbb{P}_{-i}(-j)[-2j][2](1).$$

By taking components, we have arrows

$$\tilde{e}(\varphi)_{ij}^{kl} : \mathbb{P}_{-i}(-j)[-2j] \longrightarrow \mathbb{P}_{-k}(-l)[-2l][2](1).$$

In matrix terms, we have the following lemma.

Lem 2.5. The splitting $\phi_I : \mathbb{V}_* \cong \mathbb{V}$ is characterized among the good ones by the following two conditions:

1. for $0 \leq j < i$, we have $\hat{e}(\phi_I)_{ij}^{kl} = 0$ except for $\hat{e}(\phi_I)_{ij}^{i,j+1} = \text{id}$
2. for $j = i$, we have $\hat{e}(\phi_I)_{ii}^{kl} = 0$ except, possibly, for $l \leq i$.

Using the same procedure as before, one may construct the dual $\phi_{II}(e) := ((\phi_I(e^\circ))^\circ)^{-1}$. For a detailed construction, one may refer to last paragraph in section 2.1. In general, one may find this different from ϕ_I . Roughly speaking, if we consider the arrows in the dual category as those reversed from the original category, one can imagine the construction of ϕ_{II} directly by looking for a primitive element of the highest weight and recursively looking for its preimage under e , which is just a reverse process as that of constructing ϕ_I , though we have ignored the dual version of the “appropriate” lifting f_i for simplicity here.



However, even in this rough case problem may happen, as in the example 3.1, where we indeed need not further regulate the lift f_i , since the lifted-to-top vector $e^\circ v$, where $v \in \mathbb{P}_{-r}$ may not even be annihilated by e , due to the fact the e is only nilpotent when we are considering its action on graded parts, not itself! For instance in example

3.1, $e^3(v_{-2}) = v_2 \neq 0$, hence we cannot directly take v_2 to be the starting point of the dual splitting. This may roughly lead to an intuition that it is the mixed degree part e that leads to the non-duality of the splitting, which we would summarize as a proposition below using the concept of *e-good* splitting.

Def 2.1. We say that a splitting $\varphi : \mathbb{V}_* \cong \mathbb{V}$ is *e-good* if it induces the identity on the graded pieces and $\tilde{e}(\varphi)$ is homogeneous of degree two:

$$\tilde{e}(\varphi) = \tilde{e}(\varphi)^{\{2\}}.$$

One can check algebraically that, φ is *e-good* if and only we have that

$$\hat{e}(\varphi)_{ij}^{kl} = 0 \quad \text{except for } \hat{e}(\varphi)_{ij}^{i,j+1} = \text{Id}, \quad \forall 0 \leq j \leq i-1.$$

Thus the second term in the characterization of ϕ_I leads to some trouble. By arguing more carefully by linear algebra, one would finally reach the following result that we sensed in the preceeding discussion. I hope one may find the intuition persuasive so that my omission of the proof will not cause too much question.

Prop 2.1. The first Deligne isomorphism $\phi_I(e)$ is *e-good* if and only if $\phi_I(e) = \phi_{II}(e)$.

2.3 Deligne's third decomposition ϕ_{III}

The goal now is to construct a self-dual splitting. In the previous example, one may sense that the lower degree term of the operator e itself will lead to trouble, so in the construction now we want to fix these lower degrees as more as possible. A systematic way to measure this most possibility is via the corresponding hard Lefschetz condition for “the algebra of linear maps” (indeed the algebra of arrows, in categorical terms), acted by e via its adjoint ad_e , as an upgraded version of the original (filtered) vector spaces. I think this is really an ingenious idea from Deligne, in his paper [2], and one would see how the adjoint actions fix the lower degree terms during the proof. For now, let's first construct the upgraded relative hard Lefschetz.

Formally speaking, let $\mathbb{V}, \mathbb{W}$ be in $\mathcal{AF}$, and set

$$\mathfrak{L}_{(i,j)}^{[n]}(\mathbb{V}, \mathbb{W}) := \text{Hom}_{\mathcal{AF}}(\mathbb{V}(i), \mathbb{W}(j)[n]).$$

Up to canonical isomorphism by Tate shift functors, this space above only depends on the difference $m = j - i$, and we will denote the resulting space as $\mathfrak{L}_{(m)}^{[n]}$. Now according to the degree of a map $\mathbb{V}_p \rightarrow \mathbb{V}_q[n](m)$, which is $q - p$, we can decompose $\mathfrak{L}_{(m)}^{[n]}$ further by the degree into $\bigoplus_{d=-2r}^{2r} \mathfrak{L}_{(m)}^{[n],\{d\}}$.



For notational simplicity, I will omit n and m in most places in the following discussion. The removing of n causes no trouble since as long as we fix one, it remains all the same in the discussion below, since we included the information of the degree of a map. However, one should remember that m indeed changes. For explicit index of m , one may refer to paper [1] directly.

Now we want to construct an $\mathfrak{sl}_2$ -triple on the space $\bigoplus_{d=-2r}^{2r} \mathfrak{Q}^{\{d\}}$. Recall the case as in lemma 2.3, as explained before this lemma, usually it is proved using the existence of an $\mathfrak{sl}_2$ -triple. However we have seen therein it is hard Lefschetz that implies this primitive Lefschetz decomposition in general. And indeed we have a partial inverse in characteristic 0 to the usual idea, that is, in turn to obtain an $\mathfrak{sl}_2$ -triple using an explicit primitive Lefschetz decomposition. In view of lemma 2.3, this can be said in a following way:

Lem 2.6. Suppose we have a series of vector spaces V_i and e as in lemma 2.3, with an extra requirement of V in characteristic 0 now. Also suppose we have a semisimple degree 0 operator b . Then the existence of an $\mathfrak{sl}_2$ -triple $\{e, b, f\}$ is equivalent to the hard Lefschetz condition on e^k for every k .

Proof. The implication of the hard Lefschetz condition from the $\mathfrak{sl}_2$ -triple can be found in standard representation theory texts. And we only indicate the other way round. In light of lemma 2.3, we only need to construct f using the primitive Lefschetz decomposition. The final result is that we define

$$f(e^j x) := j(-l - j + 1)e^{j-1}(x), \text{ for } x \in P_{-l}.$$

and extend this recursively and linearly into each component occurred in the primitive Lefschetz decomposition. One can verify that these indeed satisfy the condition for an $\mathfrak{sl}_2$ -triple (which is indeed easy since the coefficient is just solved in a reversed way as in calculation of eigenvalues of an $\mathfrak{sl}_2$ -module). Note the assumption on characteristic 0 ensures the coefficient before e^{j-1} will not be 0 and not cause trouble in verification. $\square$

Now in our case, the assumption of characteristic 0 is just requiring $\mathcal{A}$ to be $\mathbb{Q}$ -linear. We work under this setting (and one may find this indeed necessary in the calculation of example 3.1). Now let $b := \sum_p p \text{id}_{V_p}$, which is an degree 0 arrow in $\mathfrak{Q}$.



While for e , we do not directly use itself, and instead use its pure degree 2 part (Caveat!): we set $\bar{e} = \sum e_p$, where $e_p : V_p \rightarrow V_{p+2}(1)$ are the graded part of pure degree 2 of e , making $\bar{e}$ into an elementary in $\mathfrak{Q}^{\{2\}}$. Then they satisfy the condition of lemma 2.6 and we can in turn construct the corresponding operator f of degree -2, constituting an $\mathfrak{sl}_2$ -triple along with $\bar{e}, b$.

Now, using Jacobi identity, as in the case when adjoint of matrices make $\text{End}(V)$ into a Lie algebra, one may see that $(\bar{e}, b, f)$ is also an $\mathfrak{sl}_2$ -triple when they are acting on $\mathfrak{Q}$ via adjoint actions. For example, for $u \in \mathfrak{Q}$, $e(u)$ would always mean $\bar{e} \circ u - u \circ \bar{e}$. Hence later we would still use (e, b, f) to denote the triple, however the actual action of it should not be misunderstood. Also the hard Lefschetz condition still holds for e when acting on L adjointly. We summarize these background setting results in to the following theorem and formally include all the indices.

Thm 2.1. The arrows (b, e, f) as above form an $\mathfrak{sl}_2(\mathbb{Q})$ -triple turning the rational vector spaces

$$\mathfrak{Q}_j^{[n]} : \bigoplus_{d \in \mathbb{Z}^{\text{even/odd}}} \mathfrak{Q}_{j+d/2}^{[n], \{d\}}(\mathbb{V}_*, \mathbb{V}_*)$$

into $\mathfrak{sl}_2(\mathbb{Q})$ -modules; in what above, j is a fixed integer multiple of $1/2$ and the sum is over the integers d with fixed parity, even if j is integral, odd if j is an half-integer. For $|d| \leq 2r$, we have that the corresponding HL statement reads as follows: (e^k the k -th iteration of e)

$$e^k : \mathfrak{Q}_{(j-k/2)}^{[n],(-k)} \xrightarrow{\cong} \mathfrak{Q}_{(j+k/2)}^{[n],(k)}.$$

Now we can state the existence and uniqueness of ϕ_{III} as the following theorem.

Thm 2.2. There exists a unique good splitting, denoted by $\phi_{III}(e)$, subject to the following conditions: $e^{1-d}(\tilde{e}(\phi)^{\{d\}}) = 0$, for all $d \leq 1$.

Proof. The construction of the splitting is by recursively modifying any good splitting ϕ degree-wise to finally reach one satisfying the condition.

Let $d = 1$. The condition (80) reads $\tilde{e}^{\{1\}} = 0$. Set $\phi_1 := \phi(\text{id} + \psi^{\{-1\}})$, where $\psi^{\{-1\}} \in \mathfrak{Q}_{(0)}^{\{1\},\{-1\}}$ is a variable arrow. We conjugate e and obtain:

$$\tilde{e}(\phi_1) = (\text{id} + \psi^{\{-1\}})^{-1} \phi^{-1} e \phi (\text{id} + \psi^{\{-1\}}) \equiv \tilde{e}(\phi) + e(\phi \psi^{\{-1\}}) \pmod{\text{degree} \leq 0}.$$

Note that the last term on the left is in $\mathfrak{Q}^{\{1\}}$. We take the degree 1 part of the right hand side of the equation above and set it equal to zero:

$$e(\phi \psi^{\{-1\}}) = -\tilde{e}(\phi)^{\{1\}} \quad (\text{equality in } \mathfrak{Q}^{\{1\}}).$$

This is where we find the use of the hard Lefschetz that we have derived! The hard Lefschetz for $\mathfrak{Q}$ ensures that such a $\phi \psi^{\{-1\}}$ exists and is unique and hence determines ϕ_1 since ϕ is invertible.

Let $d = 0$. The condition becomes $e(\tilde{e}^{\{0\}}) = 0$. Set $\phi_0 := \phi_1(\text{id} + \psi^{\{-2\}})$, where $\psi^{\{-2\}} \in \mathfrak{Q}_{(0)}^{\{1\},\{-2\}}$ is a variable arrow. We conjugate e and obtain

$$\tilde{e}(\phi_0) = (\text{id} + \psi^{\{-2\}})^{-1} \phi_1^{-1} e \phi_1 (\text{id} + \psi^{\{-2\}}) \equiv \tilde{e}(\phi_1) + e(\phi_1 \psi^{\{-2\}}) \pmod{\text{degree} \leq -1}.$$

Note that the last term on the left is in $\mathfrak{Q}^{\{0\}}$. We take the degree 0 part of the right hand side of the equation above and set it equal to zero after application of e :

$$e^2(\phi_1 \psi^{\{-2\}}) = -e(\tilde{e}(\phi_1)^{\{0\}}) \quad (\text{equality in } \mathfrak{Q}^{\{2\}}).$$

Again, the hard Lefschetz for $\mathfrak{Q}$ ensures that such a $\psi^{\{-2\}}$ exists and is unique. This determines ϕ_0 .

We repeat this procedure for all decreasing values of d and since $\mathbb{V}$ has type $[-r, r]$, the procedure ends no later than $d = -2r$.

Now we check the uniqueness of the desired splitting. Suppose a, b are two good splittings satisfying the condition. Set

$$c := b^{-1}a = \text{Id} + \sum_{l \geq 1} c^{\{-l\}} : \mathbb{V}_* \xrightarrow{\cong} \mathbb{V}.$$

We apply the procedure carried out above to b , modifying it to $b_1 := b(\text{Id} + c^{\{-1\}})$. We have that $b_1 \equiv a$ modulo degree ≤ -2 , so that, in view of the fact that a also satisfies the condition for the splitting, we must have that $c^{\{-1\}} = 0$. It follows that $b \equiv a$, modulo degree ≤ -1 . We repeat this procedure and kill all the $c^{\{-l\}}$. $\square$

Rmk 2.2. To check the self-duality for ϕ_{III} , one only needs to observe that the condition in the theorem is self-dual (in the sense that if φ satisfies this condition for e , then in the dual category, $(\varphi^\circ)^{-1}$ satisfies that for e°). And one can directly calculate this, only keeping in mind that an operator becomes its transpose in the dual category, with each degree part remaining the same, and hence verifying that the adjoint action of e° is just $-e$, since ${}^t X {}^t T - {}^t T {}^t X = {}^t (TX - XT)$.

Then one calculates that:

$$({}^t e)^{1-d} (\widetilde{e}({}^t \varphi)^{-1})^{\{d\}} = {}^t (e^{1-d}) (({}^t \varphi {}^t e {}^t \varphi^{-1})^{\{d\}}) = (-1)^{1-dt} (e^{1-d} (\widetilde{e}(\varphi)^{\{d\}})).$$

Rmk 2.3. As explained in the beginning of the section, one may intuitively think of the proof as a systematic way to eliminate the low degree error using the adjoint action. From the proof we may see the adjoint action is exactly the base change if we are just looking at 1 degree lower part, modulo parts with degree even more lower. This may make the consideration of the adjoint action on the space $\mathfrak{L}$ and the upgraded hard Lefschetz seem more natural.

Also one can reformulate a characterizing condition for ϕ_{III} , using f as constructed above: $[f, \widetilde{e}(\varphi) - e] = 0$, which is included in the appendix of paper [1], a letter from Deligne to de Cataldo. The equivalence is not hard to check using the primitive Lefschetz decomposition for the space $\mathfrak{L}$ and in this way one may better perceive the usage of f to measure the error to a unique choice.

Finally, in general all the 5 splittings as constructed above are different and one may wonder when they all coincide. As a continuation of proposition 2.1, we have a stronger result stating that the case when e is of pure degree 2 is indeed the ideal case, whose proof we will omit as before.

Prop 2.2. If an e -good splitting exists, then it is unique and coincides with the third Deligne splitting ϕ_{III} , and in this case all the five splittings are the same.

3 Calculation of the decompositions in concrete examples

In this section we calculate the examples 2.6.4 and 2.6.5 in paper [1], pinpointing some crucial points that may cause confusion when we are getting our hands dirty in the real world. Not all splittings will be calculated out fully, only relations between certain splittings spoken in some examples by looking at the properties of the matrix of operator, since I've already included a heavy amount of calculation and I want omit some if enough to explain the author's idea in paper [1], also because some relations is easy to read without calculating everything such as $\kappa \neq \phi_{III}$ in example 3.2.1.

In fact, ϕ_{III} 's in example 3.2 are much easier so the really exemplifying calculation of it is in example 3.1, while that of ω_I and ω_{II} is in example 3.2.2 (though in example 3.1 it is already not so easy to make things all correct, as contrast to its most naive construction idea in abstract setting). ϕ_I is seldom calculated carefully since in most of the examples the primitive Lefschetz decomposition is simple enough to allow us make little modification to get ϕ_I , so as ϕ_{II} , thus you may find the calculation and explanation of these to be (too) concise.

3.1 Example of a vector space

Let $V \cong \mathbb{Q}^3$ with basis (v_{-2}, v_0, v_2) , $e : (v_{-2}, v_0, v_2) \mapsto (v_0, v_2, v_2)$,

$$V_{\leq -2} = V_{\leq -1} = \langle v_{-2} \rangle, \quad V_{\leq 0} = V_{\leq 1} = \langle v_{-2}, v_0 \rangle, \quad V_{\leq 2} = V,$$

$$V_{-2} = \langle [v_{-2}] \rangle, \quad V_0 = \langle [v_0] \rangle, \quad V_2 = \langle [v_2] \rangle.$$

The five splittings associated with e discussed in this paper are:

- $\phi_I(e) : ([v_{-2}], [v_0], [v_2]) \mapsto (v_{-2}, v_0, v_2)$;
- $\phi_{II}(e) : ([v_{-2}], [v_0], [v_2]) \mapsto (v_{-2}, -v_{-2} + v_0, -v_0 + v_2)$;
- $\phi_{III}(e) : ([v_{-2}], [v_0], [v_2]) \mapsto (v_{-2}, -\frac{1}{3}v_{-2} + v_0, -\frac{2}{3}v_0 + v_2)$;
- $\omega_I(e) : ([v_{-2}], [v_0], [v_2]) \mapsto (v_{-2}, -v_{-2} + v_0, v_2)$;
- $\omega_{II}(e) : ([v_{-2}], [v_0], [v_2]) \mapsto (v_{-2}, -v_{-2} + v_0, -v_{-2} + v_2)$.

In particular, using proposition 2.2, one directly finds that there is no e -good splitting.

We first calculate ϕ_I . As is explained in the first paragraph of remark 2.1, notice that P_0 is 0 in this case and the primitive Lefschetz decomposition is simply given by $V_{-2} \oplus e(V_{-2}) \oplus e^2(V_{-2})$, and the original lifting $[v_{-2}] \mapsto v_{-2}$ is already fine for V_{-2} , hence the splitting is simply given by $v_{-2}, e(v_{-2}), e^2(v_{-2})$, i.e., $\phi_I(e) = (v_{-2}, v_0, v_2)$.

Now we calculate ϕ_{II} by passing to the dual category and back. Denote the dual vectors by v_i^* , then v_2^* has the lowest weight in the dual category, and for the dual operator e° , it satisfies that

$$e^\circ(v_2^*) = v_0^* + v_2^*, \quad e^\circ(v_0^*) = v_{-2}^*, \quad e^\circ(v_{-2}^*) = 0.$$

Again one checks that the primitive part P_0 is 0 in the dual case. Hence the splitting in the dual category is again given by $v_2^*, e(v_2^*), e^2(v_2^*)$.



We now write the splitting basis in the dual category, in a “reversed” way that the first component of each column corresponding to the highest weight vector and the weight decreases from the top to the bottom, i.e., the first component of each column represents v_{-2}^* in our convention here. This would lead to a reversed reading order from right to left in the final matrix obtained. (I suspect that I’ve made some mistakes so these would not perfectly

match but the rows and columns is really cumbersome to deal with so I give up to find a better formulation.) So now the matrix becomes

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Taking the transpose and inverse of it, we get

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Reading from right to left as explained before, we get

$$\phi_{II}(e) : ([v_{-2}], [v_0], [v_2]) \mapsto (v_{-2}, -v_{-2} + v_0, -v_0 + v_2).$$

Now we first calculate ω_I using the procedure as in section 2.1. We can explicitly write out now

$$p_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad i_{-2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence using the formula for the left inverse in section 1.1., we get

$$\rho = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Its kernel given by a vector $v_0 - v_{-2}$, together with v_{-2}, v_2 , we get the desired splitting

$$\omega_I(e) : ([v_{-2}], [v_0], [v_2]) \mapsto (v_{-2}, -v_{-2} + v_0, v_2).$$



For ω_{II} , we pass to the dual category. But note that now the highest vector, to be denoted by u_{-2}^* for the splitting ω_{II} is not simply v_{-2}^* itself since we would require $(e^\circ)^2$ induce an isomorphism from V_2^* to V_{-2}^* . Hence the natural choice is to take it as $(e^\circ)^2(v_2^*) = v_{-2}^* + v_0^* + v_2^*$. Bearing this and the convention for matrix notation as explained before, we first calculate the corresponding ρ° for u_i^* , which is equal to

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Its kernel given by v_0^* , together with v_2^* and $v_{-2}^* + v_0^* + v_2^*$, we get a splitting matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Taking the transpose and inverse of it, we get

$$\begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Reading from right to left as explained before, we get

$$\omega_{II}(e) : ([v_{-2}], [v_0], [v_2]) \mapsto (v_{-2}, -v_{-2} + v_0, -v_{-2} + v_2).$$

Finally let us calculate ϕ_{III} . Recall that for this, we need to use the adjoint action of the degree 2 part $\bar{e}$ of e , which changes the element $e_{3,3}$ to 0, eliminating this degree error with other terms remaining the same. Also notice that there is no degree 1 terms. So the first effective condition becomes that $[\bar{e}, X] = 0$, where X is the degree 0 part of e , which is simply a matrix with $X_{3,3} = 1$ and other terms 0. Thus using the method as in the proof of theorem 2.2, we conjugate ϕ_I by a matrix ψ of pure degree -2, supposed to be

$$\begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

Then one can calculate that

$$[\bar{e}, [\bar{e}, \psi]] = \begin{pmatrix} 0 & 0 & 0 \\ y - 2x & 0 & 0 \\ 0 & x - 2y & 0 \end{pmatrix},$$

which should be equal to

$$[\bar{e}, X] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

leading to $x = -\frac{1}{3}, y = -\frac{2}{3}$. One can further verify that further conditions for $\tilde{e}^{\{2\}}$ and $\tilde{e}^{\{4\}}$ are already satisfied.

Hence we have obtained the desired splitting:

$$\phi_{III}(e) : ([v_{-2}], [v_0], [v_2]) \mapsto (v_{-2}, -\frac{1}{3}v_{-2} + v_0, -\frac{2}{3}v_{-2} + v_2).$$

3.2 Example of (mixed) Hodge structures

Let $Y \times Z$ be the product of nonsingular complex projective varieties and let $r := \dim_{\mathbb{C}} Z$. Let

$$V := H(Y \times Z, \mathbb{Q}) = H(Y, \mathbb{Q}) \otimes H(Z, \mathbb{Q}) = \bigoplus_{r,s} H^r(Y, \mathbb{Q}) \otimes H^s(Z, \mathbb{Q}).$$

Let $F'_s V$ be the subspace spanned by the elements of the form $y_i z_\sigma$ with $\sigma \leq s$, and $i = 0, 1, \dots, r$. (Note that this filtration is not given by the total degree of $H^k(Y \times Z)$, rather, it is given by the degree of the $H^k(Z)$ part.)

Set $F := F'[r]$ so that (V, F) has type $[-r, r]$. Let $\eta \in H^2(Y \times Z, \mathbb{Q})$ be the first Chern class of a line bundle on $Y \times Z$ which is ample when restricted to the fibers of the projection onto Y . Denote by $e : (V, F) \rightarrow (V, F[2])$ the map $v \mapsto \eta \cup v$. By using the hard Lefschetz theorem on Z , one sees directly that the HL condition holds for $((V, F), e)$. Besides the five splittings considered in this paper, we also have the Künneth splitting κ . This is the analogue to the most “straightforward” splitting as ϕ_I in this previous example, i.e., each graded part of it is generated by pure tensor products $y_i \otimes z_\sigma$ where $y_i \in H^i(Y)$, $z_\sigma \in H^\sigma(Z)$. In general, the six splittings are pairwise distinct.

3.2.1 Elliptic curves and Poincaré bundle

First we consider the example by taking $Y = Z$ to be an elliptic curve E and the line bundle to be of the form $E \times \zeta + \zeta \times E + P$, where $\zeta \in E$ is a point and P is a Poincaré bundle. (More precisely in symbol, $E \times \zeta$ is a divisor and what we really means here is its associated line bundle, and same for $\zeta \times E$.) It is exactly the existence of this part of Poincaré bundle that ensures $\tilde{e}(\kappa)^{\{1\}} \neq 0$, so that by definition, we see that $\kappa \neq \phi_{III}(e)$. This is because if we pass to cohomology, this term would lead to a cohomology class $\gamma \in H^1(E) \otimes H^1(E)$, while the other two terms lead to $1 \otimes u + u \otimes 1$, where $u = c_1(\zeta) \in H^2(E)$. Hence we can directly calculate the corresponding matrix of this splitting and we find that $\tilde{e}(\kappa)^{\{1\}} = \gamma \text{Id}_2 \neq 0$. (We would not calculate the explicit formula for ϕ_{III} here due to length constraint.)

3.2.2 Product of projective spaces

Now we take $Y = \mathbb{P}^1, Z = \mathbb{P}^2$, we are lead to examples where $\kappa = \phi_{III}(e)$, but otherwise the isomorphisms of type ϕ and ω are pairwise distinct, and distinct from $\phi_{III}(e)$. Let's now calculate this example.

Denote the pure tensor of class $b^i \otimes k^j$ by $v_{2j-2,2i}$, where $i = 0, 1$ and $j = 0, 1, 2$, and b, k are the hyperplane class of $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively. For example, $1 \otimes 1 =: v_{-2,0}$ while $b \otimes k =: v_{0,2}$. The shift is to keep in concordance with shifted cohomology degrees to ensure symmetry around 0'th degree. Still, $v_{s,t}$ has degree $s+t$ in the unshifted cohomology. In this notation, the three graded parts $V_i, i = -2, 0, 2$ are generated by $v_{i,0}$ and $v_{i,2}$ and one can verify the operator $e = 1 \otimes k + b \otimes 1$ has the following matrix form with respect to the κ splitting:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \textcolor{teal}{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \textcolor{teal}{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & \textcolor{teal}{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \textcolor{teal}{1} & 1 & 0 \end{pmatrix},$$

where the degree 2 part is marked by **teal color** while the degree 0 term is marked by normal black color.



Note that in this example we do not simply deduce the degree of $e_{p,q}$ by $p - q$, i.e., which diagonal it is on, instead this is determined by which 2×2 block it is in.

Let's first observe that $\kappa = \phi_{III}$, this has no degree 1 part (in contrast with the example of elliptic curves, which has an extra error term given by Poincaré bundle), and no degree i parts with $i < 0$. Also, one can check that the degree 0 part commute with the whole, hence with the degree 2 part, thus checking $\kappa = \phi_{III}$. As in the example 3.1 of vector space, here ϕ_I is direct to see since again $P_0 = \ker e \cap V_0 = 0$, hence the splitting ϕ_I is simply given by iterating the basis from P_{-2} to $e(P_{-2}), e^2(P_{-2})$, leading to the basis consisting of

$$\phi_I(e) = (v_{-2,0}, v_{-2,2}, v_{-2,2} + v_{0,0}, v_{0,2}, 2v_{0,2} + v_{2,0}, v_{2,2}).$$

Now let's compute ω_I and ω_{II} . For ω_I , as in example 3.1, we have the following matrices:

$$p_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \end{pmatrix}, \quad i_{-2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence using the procedure in section 2.1, one can directly calculate a left inverse of $i_{-2} + e^2 i_{-2} (e^2_{-2})^{-1}$ (where $(e^2_{-2})^{-1}$ is simply the 2×2 identity matrix), which is

$$\rho = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

whose kernel consists of $2v_{-2,2} - v_{0,0}$ and $v_{0,2}$, leading to a full set of splitting basis

$$\omega_I(e) = (v_{-2,0}, v_{-2,2}, 2v_{-2,2} - v_{0,0}, v_{0,2}, v_{2,0}, v_{2,2}).$$

Now for $\omega_{II}(e)$, as explained in example 3.1, the matrix e^2 is replaced by its transpose in the dual case while the identity part of p_2 goes to the first two column, and dually that of i_{-2} goes to the last column, leading to the left inverse

$$\rho^\circ = \rho = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$



Caveat again that the columns in this matrix corresponds in an reversed way to $u_6^*, u_5^*, u_4^*, u_3^*, u_2^*, u_1^*$! Here u_i^* are

determined by the dual filtration induced by e° , i.e.,

$$(u_6^*, u_5^*, u_4^*, u_3^*, u_2^*, u_1^*) = (v_6^*, v_5^*, e(v_6^*), e(v_5^*), e^2(v_6^*), e^2(v_5^*)),$$

where v_i^* is a set of dual basis of v_i (now in lexicographic order to write the index from 1 to 6). Substituting u_i^* by v_i^* into the basis $(u_6^*, u_5^*, u_4^* - 2u_5^*, u_3^*, u_2^*, u_1^*)$ given by the above matrix ρ° , we get the following matrix A giving the splitting basis in terms of v_i^* and its inverse:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As explained in example 3.1, reading as above from right to left, we have finished the calculation of

$$\omega_{II}(e) = (v_{-2,0}, v_{-2,2}, v_{-2,2} + v_{0,0}, v_{0,2}, v_{2,0} - 2v_{0,2}, v_{2,2}).$$

Till now, we have already checked very explicitly that $\omega_I \neq \phi_I$ and that ω_I is not self-dual, also $\phi_I \neq \omega_{II}$, hence there no need to calculate ϕ_{II} to check that these are pairwise distinct, hence I would not calculate ϕ_{II} here.

3.2.3 Another product of projective spaces

Finally we take $Y = Z = \mathbb{P}^1$, then we have $\kappa = \phi_{III}(e)$, $\phi_I(e) = \omega_I(e)$ and $\phi_{II}(e) = \omega_{II}(e)$, but we have no further relation. This example is indeed much easier, since it has only two graded parts V_{-1}, V_1 , so in particular we may see that $\phi_I = \omega_I$ by the first paragraph in remark 2.1, and as a consequence we also have $\phi_{II} = \omega_{II}$. Now to check $\phi_{III} = \kappa$ and ω_I is not self-dual, we may calculate the matrix of e as usual and carry out the explicit computation: one can directly verify that

$$e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Hence in particular, as the example before, this has no degree 1 part and degree i parts with $i < 0$, and one can check as the previous example that the degree 0 part commute with the whole, hence with the degree 2 part, thus checking $\kappa = \phi_{III}$. On the other hand, since there are only two graded parts V_{-1}, V_1 , ω_I is trivially given by the basis $v_{-1,-1}, v_{-1,1}$ and $e(v_{-1,-1}) = v_{-1,1} + v_{1,-1}$, $e(v_{-1,1}) = v_{1,1}$ (where the notation goes the same as in the previous example), constituting the basis of splitting ω_I . As in the previous example, the basis of the dual splitting ω is read

from right to the left of the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

hence $v_{-1,-1}, v_{-1,1}, -v_{-1,1} + v_{1,-1}, v_{1,1}$, thus verifying ω_I is not self-dual.

References

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