

GAGA Reading Report

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0 Introduction

This article, devoted to understand some details and summarize the proof ideas of Serre's famous GAGA paper, is mainly divided into two parts. In the first part, we will give the definition of an analytic space, as well as some preparatory conclusions locally. This part is mainly to prove Corollary 1 of Proposition 3 in the original GAGA text, namely that $(\mathcal{O}_x, \mathcal{H}_x)$ constitutes a flat pair. (In other words, $\mathcal{O}_x$ can be regarded as a subring of $\mathcal{H}_x$, and $\mathcal{H}_x/\mathcal{O}_x$ is a flat $\mathcal{O}_x$ -module.) Since this conclusion will play an essential role in many places in the following text (at least it guarantees that the analytification functor $(-)^b$ to be defined below is exact), we will give a detailed proof of this conclusion in this article. To this end, we will first supplement some required more basic results beforehand.

In the second part, we will state the main theorem and give an outline of its proof. Since I think the proof in the original text is written very clearly, and the methods used in the proof are very standard, completely similar to the methods when calculating the cohomology of schemes, most of the proof details will be omitted. Finally, we give some applications of the main theorem.

1 Basic Definitions

Definition 1.1. For a subset U of $\mathbb{C}^n$, if for every point $x \in U$, there exists a neighborhood W of x and a family of holomorphic functions $f_1, \dots, f_k$ on W , such that $U \cap W = \{z \in W \mid f_i(z) = 0, i = 1, \dots, k\}$, then U is said to be analytic.

Let $\mathcal{C}(X)$ denote the sheaf of germs of complex-valued functions on a space X , and let $\mathcal{H}$ denote the sheaf of germs of holomorphic functions on $\mathbb{C}^n$. For a subset U in $\mathbb{C}^n$, at each point x in U there is a natural restriction homomorphism $\epsilon_x : \mathcal{C}(\mathbb{C}^n)_x \rightarrow \mathcal{C}(U)_x$, then let the image of $\mathcal{H}_x$ under ϵ_x be denoted as $\mathcal{H}_{x,U}$. The sheaf $\mathcal{H}_U$ formed by $\mathcal{H}_{x,U}$ is called the sheaf of germs of holomorphic functions on U , and denote $\mathcal{A}_x(U) = \ker \epsilon_x$. Then it is not difficult to see that $\mathcal{H}_{x,U} \cong \mathcal{H}_x / \mathcal{A}_x(U)$.

Definition 1.2. For an analytic subset U in $\mathbb{C}^r$ and an analytic subset V in $\mathbb{C}^s$, consider a mapping $\phi : U \rightarrow V$. If ϕ is continuous and for any $f \in \mathcal{H}_{\phi(x),V}$, we always have $f \circ \phi \in \mathcal{H}_{x,U}$ (that is, for $x \in U$, each coordinate of $\phi(x)$ is a holomorphic function of x), then f is said to be holomorphic. If a bijection ϕ satisfies that both ϕ and ϕ^{-1} are holomorphic, then ϕ is called an (analytic) isomorphism.

Next, we give the definition of a general analytic space:

Definition 1.3. An analytic space is a topological space X and a subsheaf $\mathcal{H}_X$ of $\mathcal{C}(X)$, satisfying the following two axioms:

- (1) There exists an open cover $\{V_i\}$ of X , such that each V_i (equipped with the subspace topology induced by X and the subsheaf induced by $\mathcal{H}_X$) is isomorphic to an analytic subset U_i in an affine space (the topology of U_i and the sheaf on it are as defined previously).
- (2) The topology of X is Hausdorff.

If X is an analytic space, the sheaf $\mathcal{H}_X$ in the definition is called the sheaf of germs of holomorphic functions on X . We call an $\mathcal{H}_X$ -module sheaf on X an analytic sheaf.

In the following sections, for an analytic space X and a closed analytic subspace Y therein, define the sheaf $\mathcal{A}(Y)$: for an open set $U \subseteq X$, $\mathcal{A}(Y)(U) = \{f \in \mathcal{H}_X(U) \mid \overline{f_x} = 0 \in k(x), x \in U \cap Y\}$; and correspondingly, for an algebraic variety X and a closed subvariety Y therein, define the sheaf $\mathcal{F}(Y)$: for an open set $U \subseteq X$, $\mathcal{F}(Y)(U) = \{g \in \mathcal{O}_X(U) \mid \overline{g_x} = 0 \in k(x), x \in U \cap Y\}$ (equivalent to the definition in the original text). In fact, both $\mathcal{H}_X$ and $\mathcal{A}(Y)$ are coherent sheaves, the proof of which is omitted here (this is a theorem by Cartan); in particular, $\mathcal{H}_Y \cong \mathcal{H}_X / \mathcal{A}(Y)$ is also a coherent sheaf.

Morphisms in general analytic spaces can completely copy the previous definition:

Definition 1.4. For analytic spaces X, Y , consider a mapping $\phi : X \rightarrow Y$. If ϕ is continuous and for any $f \in \mathcal{H}_{\phi(x),Y}$, we always have $f \circ \phi \in \mathcal{H}_{x,X}$ (that is, for $x \in X$, each coordinate of $\phi(x)$ is a holomorphic function of x), then f is called a morphism between these two analytic spaces. If a bijection ϕ satisfies that both ϕ and ϕ^{-1} are holomorphic, then ϕ is called an (analytic) isomorphism.

Definition 1.5. For an open set V in an analytic space X , if there exists an analytic isomorphism $\phi : V \rightarrow U$, where U is an analytic subset in some affine space, then this isomorphism is called an (analytic) chart of V , and the first axiom is thus requiring that X can be covered by open sets with charts. For a subset Y in X , if for every chart $\phi : V \rightarrow U$, $\phi(Y \cap V)$ is an analytic subset in U , then Y is said to be analytic, and it has a natural analytic space structure, called the analytic structure induced by X .

Similarly, for a Zariski open set V in an algebraic variety X , if there exists an algebraic isomorphism $\phi : V \rightarrow U$, where U is a locally closed set in some affine space, then this isomorphism is called an (algebraic) chart of V .

After making these preparations, we can define the analytic space associated with an algebraic variety. To this end, note that polynomials are all holomorphic functions, and more naturally continuous functions, thus the

Zariski topology on $\mathbb{C}^n$ is strictly coarser than the classical topology on it, and the locally closed sets under the Zariski topology are naturally also analytic subsets. At the mapping level, regular mappings between two Zariski locally closed sets are naturally also holomorphic mappings; algebraic isomorphisms are naturally also analytic isomorphisms. Then the existence and uniqueness of the analytic space X^b associated with an algebraic variety X are guaranteed by the following theorem:

Theorem 1.1. An algebraic variety X has a unique analytic space structure X^b , such that for each (algebraic) chart $\phi : V \rightarrow U$, the Zariski open set V is also an open set (under the classical topology), and ϕ is an analytic isomorphism. (In other words, every algebraic chart is an analytic chart).

The proof of this theorem can be achieved simply by repeatedly applying the aforementioned comparison between the Zariski topology and the general topology. In particular, it is not difficult to prove that the analytic space structure defined in this way indeed satisfies the Hausdorff property; one only needs to start from the separatedness of the algebraic variety and do similar work. From a topological perspective, the topology on the analytic space is still finer than the Zariski topology of the algebraic variety; on the other hand, analytic spaces can be understood as generalizations of complex manifolds, and compared with general real smooth manifolds, holomorphicity brings them a certain "rigidity", therefore some kind of comparison can be made with algebraic varieties. (This "rigidity" can more or less be glimpsed from the Cauchy integral formula, and as a corollary of this, we can notice the equivalence of holomorphicity and analyticity on the complex plane. Then comparing polynomials and formal power series, from an algebraic point of view, I think this implies the results regarding flatness that we will focus on below.)

2 Extra Preparatory Results

In this part, we will sort out and list some conclusions in the appendix of the original text or cited in the original text, in order to prove Theorem 2.3. Then first we give the definition of a flat pair and an equivalent characterization.

Definition 2.1. For two rings $A \subseteq B$, if B/A is a flat A -module, then (A, B) is said to constitute a flat pair.

In fact, what this definition requires is nothing more than two aspects, namely the following theorem:

Theorem 2.1. For rings $A \subseteq B$, (A, B) is a flat pair if and only if B is a flat A -module, and for any (finitely generated) module E , the homomorphism $E \mapsto E \otimes_A B$ is injective.

Its proof only needs to consider the long exact sequence induced by the short exact sequence $0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0$, and note that A is flat as a module over itself.

The most important example of a flat pair is a Noetherian local ring and its completion:

Theorem 2.2. A Noetherian local ring A and its completion constitute a flat pair $(A, \hat{A})$.

In fact, for a general ($\mathfrak{a}$ -adic) completion homomorphism $\phi : M \mapsto \hat{M}$, by Krull's theorem, $\ker \phi = \text{Ann}(1 + \mathfrak{a})$. As a corollary, for modules over a Noetherian local ring A , this completion homomorphism is injective, see [AM] p.110. On the other hand, for a finitely generated A -module M , the functor $M \mapsto \hat{M}$ is an exact functor (but not so for non-finitely generated rings), and in general the canonical homomorphism $\hat{A} \otimes_A M \rightarrow \hat{M}$ is not an isomorphism; it is only so under the assumption of Noetherianness. Therefore, when A is a Noetherian ring, since the criterion of flatness in fact only needs to consider finitely generated modules, we know that $\hat{A}$ is flat over A , see [AM] p.109. Combining these two aspects can prove Theorem 2.2.

Theorem 2.3. For rings $A \subseteq B \subseteq C$, if both (A, C) and (B, C) are flat pairs, then (A, B) is as well.

Applying these two theorems above can give the criterion we ultimately need:

Theorem 2.4. For two Noetherian local rings A, B , let their maximal ideals be $\mathfrak{m}, \mathfrak{n}$ respectively. If a homomorphism $\theta : A \rightarrow B$ satisfies $\theta(\mathfrak{m}) \subseteq \mathfrak{n}$, then it naturally induces a homomorphism $\hat{\theta} : \hat{A} \rightarrow \hat{B}$ between their completions. If θ is injective (i.e. makes A can be regarded as a subring of B through its action) and $\hat{\theta}$ is bijective, then (A, B) is a flat pair.

It should be noted that in the conditions of this theorem, if $\hat{\theta}$ is a bijection, we can consider $\ker \theta$. By the exactness of $\otimes_A \hat{A}$ at this time, it is seen that $\ker \theta = \ker \hat{\theta} = 0$; however, since $M \mapsto \hat{M}$ is injective at this time, we see that $\ker \theta = 0$, thus θ is automatically injective. Therefore, after the proof of Theorem 3.1 is completed, as a corollary we can obtain that $\theta : \hat{\mathcal{O}}_x \rightarrow \hat{\mathcal{H}}_x$ is injective.

Theorem 2.5. For two Noetherian local rings A, B , let their maximal ideals be $\mathfrak{m}, \mathfrak{n}$ respectively. If a homomorphism $\theta : A \rightarrow B$ satisfies the conditions in Theorem 2.4, then the homomorphism $\bar{\theta} : A/\mathfrak{m}A \rightarrow B/\theta(\mathfrak{m})B$ it induces also does so.

This is because the injectivity of θ naturally induces the injectivity of $\bar{\theta}$, thus as an A -module, $\mathfrak{m} \cong \theta(\mathfrak{m})$. And from what was mentioned earlier, $A/\hat{\mathfrak{m}}A = \hat{A}/\mathfrak{m}\hat{A} \cong \hat{B}/\theta(\mathfrak{m})\hat{B} = B/\theta(\mathfrak{m})B$, which proves this theorem.

In addition to these algebraic conclusions above, for a general complex analytic space, the basic result we also need to use is the following analytic version of the Nullstellensatz, whose proof can be completed by redeveloping conclusions similar to Noetherian normalization in the analytic case (e.g. see [D] p.97), which is omitted here:

Theorem 2.6. Consider the analytic function germ $\mathcal{H}_x$ at some point x in the complex affine space $\mathbb{C}^n$. For any ideal $\mathfrak{a}$ therein, it holds that $\mathcal{A}_x(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Finally, in the proof of Theorem 3.2 below, we need to directly use the following Chevalley's theorem, from [C] p.11. Its proof is rather complicated, so we omit it here, and we also omit the definition of a "geometric local ring"; we just need to assume this theorem can be applied.

Theorem 2.7. If $\mathfrak{o}$ is a geometric local ring, for a prime ideal $\mathfrak{p}$ therein, $\mathfrak{p}\bar{\mathfrak{o}}$ is the intersection of several prime ideals in $\bar{\mathfrak{o}}$.

3 Flatness

Since every regular function is holomorphic, every regular function germ $f \in \mathcal{O}_x$ automatically defines a holomorphic function germ, denoted as $\theta(f) \in \mathcal{H}_x$. Obviously, the θ defined in this way is a ring homomorphism, and maps the maximal ideal of $\mathcal{O}_x$ into the maximal ideal of $\mathcal{H}_x$, thereby inducing a homomorphism between their completions $\hat{\theta} : \hat{\mathcal{O}}_x \rightarrow \hat{\mathcal{H}}_x$.

In this section, we will prove the following two basic propositions:

Theorem 3.1. The homomorphism $\hat{\theta} : \hat{\mathcal{O}}_x \rightarrow \hat{\mathcal{H}}_x$ is a bijection.

Theorem 3.2. The ideal $\mathcal{A}_x(Y)$ is generated by $\theta(\mathcal{J}_x(Y))$.

Proof. We first prove these two theorems in the affine case. Theorem 3.1 is obvious, because for an affine space $X = \mathbb{C}^n$, $\mathcal{H}_x$ is exactly the algebra $\mathbb{C}\{z_1, \dots, z_n\}$ formed by all convergent n -variable power series, thus its completion is isomorphic to the formal power series ring $\mathbb{C}[[z_1, \dots, z_n]]$, whereas $\mathcal{O}_x \cong \mathbb{C}[z_1, \dots, z_n]_{(z_1, \dots, z_n)}$, thus its completion is also the formal power series ring. Now we prove Theorem 3.2. Let $\mathfrak{a} = (\theta(\mathcal{J}_x(Y)))$. Then it is not difficult to see that $V(\mathfrak{a}) = Y$, so by the analytic version of the Nullstellensatz mentioned above, for $f \in \mathcal{A}_x(Y)$, there exists a positive integer r such that $f^r \in \mathfrak{a}$, thus naturally we have $f^r \in \mathfrak{a}\hat{\mathcal{H}}_x = \mathcal{J}_x(Y)\hat{\mathcal{H}}_x = \mathcal{J}_x(Y)\hat{\mathcal{O}}_x$.

Then in the algebraic variety version we know that $\mathcal{J}_x(Y)$ is the intersection of prime ideals (corresponding to the irreducible components passing through x), and by Chevalley's theorem given above, we know that $\mathcal{J}_x(Y)\hat{\mathcal{O}}_x$ is as well, then $f^r \in \mathcal{J}_x(Y)\hat{\mathcal{O}}_x$ implies $f \in \mathcal{J}_x(Y)\hat{\mathcal{O}}_x$. In addition, by Theorem 2.1 we know that for any module E , $E \otimes_{\mathcal{H}_x} \hat{\mathcal{H}}_x$ is injective. Then for an ideal $\mathfrak{a}$ in $\mathcal{H}_x$, we see that $\mathfrak{a} \otimes_{\mathcal{H}_x} \hat{\mathcal{H}}_x \cap \mathcal{H}_x = \mathfrak{a}\hat{\mathcal{H}}_x \cap \mathcal{H}_x = \mathfrak{a}$, thus $f \in \mathfrak{a}$, which proves Theorem 3.2 in the affine case.

Next we handle the general case. Assume without loss of generality that X is an algebraic subvariety in an affine space U , then $\mathcal{O}_{x,X} = \mathcal{O}_{x,U}/\mathcal{J}_x(X, U)$ and similarly, $\mathcal{H}_{x,X} = \mathcal{H}_{x,U}/\mathcal{A}_x(X, U)$. Then since $\theta : \mathcal{O}_x \rightarrow \mathcal{H}_x$ is naturally induced by the homomorphism $\theta_{X,U}(\mathcal{J}_x(X, U)) \subseteq \mathcal{A}_x(X, U)$, and from the discussion of the affine case earlier we know that $\hat{\theta}_{X,U}$ is a bijection, and $\mathcal{A}_x(X, U) = \theta(\mathcal{J}_x(X, U))\mathcal{H}_{x,U}$, then by Theorem 2.5 we have proved Theorem 3.1 in the general case. In addition, note that $\mathcal{A}_x(Y)$ is exactly the canonical image of $\mathcal{A}_x(Y, U)$ under the quotient mapping, and by the affine case, the latter is generated by $\theta(\mathcal{J}_x(Y, U))$, which proves Theorem 3.2. $\square$

Then as a corollary, we obtain the final theorem:

Theorem 3.3. $(\mathcal{O}_x, \mathcal{H}_x)$ constitutes a flat pair.

4 Statement of the Main Theorem

We first define an analytic sheaf $\mathcal{F}^b$ associated with a sheaf $\mathcal{F}$ on an algebraic variety X . First, we assign a new topology to the set $\mathcal{F}$ to make it a sheaf on X^b . To this end, first consider the projection $\pi : \mathcal{F} \rightarrow X$, and thereby obtain an embedding $\mathcal{F} \rightarrow X^b \times \mathcal{F}, f \mapsto (\pi(f), f)$. Then by taking the subspace topology induced on $\mathcal{F}$ by the product topology of $X^b \times \mathcal{F}$, we obtain a sheaf $\mathcal{F}'$ on X^b . Then $\mathcal{F}'$ and $\mathcal{F}$ only differ in topological structure; at least at each point x , we have $\mathcal{F}'_x = \mathcal{F}_x$.

In particular, for the regular function sheaf $\mathcal{O}$, we can consider the corresponding $\mathcal{O}'$, then both $\mathcal{F}'$ and $\mathcal{H}$ are $\mathcal{O}'$ -module sheaves, so we can define $\mathcal{F}^b = \mathcal{F}' \otimes_{\mathcal{O}'} \mathcal{H}$ as the analytic sheaf associated with $\mathcal{F}$. And canonically, for each algebraic sheaf morphism $\phi : \mathcal{F} \rightarrow \mathcal{G}$, we can define its associated analytic sheaf morphism $\phi^b : \mathcal{F}^b \rightarrow \mathcal{G}^b$. In this way, we have obtained an analytification functor.

Since $(\mathcal{O}_x, \mathcal{H}_x)$ constitutes a flat pair, we can obtain some basic properties of the analytification functor $(-)^b$. We know that $\otimes_{\mathcal{O}_x} \mathcal{H}_x$ is exact, thus the analytification functor is exact; and for any coherent sheaf $\mathcal{F}$, $\mathcal{F}_x \rightarrow \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{H}_x$ is always injective, thus $\mathcal{F}' \rightarrow \mathcal{F}^b$ is always injective. By exactness, we can also easily prove directly from the definition that for a coherent sheaf $\mathcal{F}$, $\mathcal{F}^b$ is still a coherent sheaf. In a word, the functor $(-)^b$ obtained according to the above construction is an exact functor $\mathbf{Coh}(X) \rightarrow \mathbf{Coh}(X^b)$.

Then for each open set U in X and a section s on it, via the mapping $s \mapsto s \otimes 1$, we obtain a canonical homomorphism $\varepsilon : \Gamma(U, \mathcal{F}) \rightarrow \Gamma(U^b, \mathcal{F}^b)$. Similarly, we can obtain a canonical homomorphism between Čech complexes $\varepsilon : C(\mathcal{U}, \mathcal{F}) \rightarrow C(\mathcal{U}^b, \mathcal{F}^b)$, thereby inducing a homomorphism between cohomology groups, still denoted as $\varepsilon : H^q(\mathcal{U}, \mathcal{F}) \rightarrow H^q(\mathcal{U}^b, \mathcal{F}^b)$.

In the following, X will always represent a complex projective algebraic variety, that is, a closed subvariety in the projective space $\mathbb{P}^r(\mathbb{C})$. Next, we state the main theorem, which contains the following three parts:

Theorem 4.1. For every coherent sheaf $\mathcal{F}$ on X and all $q \geq 0$, $\varepsilon : H^q(X, \mathcal{F}) \rightarrow H^q(X^b, \mathcal{F}^b)$ defined above is a bijection.

Theorem 4.2. If $\mathcal{F}, \mathcal{G}$ are coherent sheaves on X , for any sheaf morphism $g : \mathcal{F}^b \rightarrow \mathcal{G}^b$, there exists a unique sheaf morphism $f : \mathcal{F} \rightarrow \mathcal{G}$, such that $g = f^b$.

Theorem 4.3. For each analytic coherent sheaf $\mathcal{M}$ on X^b , there exists an algebraic coherent sheaf $\mathcal{F}$ on X , such that $\mathcal{M} \simeq \mathcal{F}^b$, and the $\mathcal{F}$ satisfying this condition is unique up to isomorphism.

We can describe the latter two theorems using the language of category theory. Then for the analytification functor $(-)^b$, the second theorem implies that this is a fully faithful functor, while the third theorem implies that this is an essentially surjective functor (and satisfies the uniqueness of the "preimage"). Therefore, these two theorems are in fact saying that $(-)^b$ gives an equivalence between these two categories $(-)^b : \mathbf{Coh}(X) \rightarrow \mathbf{Coh}(X^b)$.

Remark 4.1. It is necessary to note that, just as the author gave in Remark 2 of the main theorem in the original text, the first part of the main theorem is not just comparing the cohomologies of two different topological spaces; the difference in sheaves is also essential: when computing the cohomology on an analytic space, the corresponding sheaf must also be replaced by the corresponding analytic sheaf in order to get the correct result. We will next give a different example and specific calculations.

In fact, for any space that can be triangulated, the Čech cohomology corresponding to a locally constant sheaf is the same as the singular homology of this space (see [Mu], p.437). Then for a compact Riemann surface of genus g , we know that $\check{H}^1(X, \underline{\mathbb{Z}}) = H_{\text{sing}}^1(X, \mathbb{Z}) = \mathbb{Z}^{2g}$, but on the other hand, we have the following result (see [Mi], p.313):

Theorem 4.4. For a compact Riemann surface X (using its Zariski topology) and any constant sheaf $\underline{G}$ (formed by an Abelian group) on it, we always have $\check{H}^n(X, \underline{G}) = 0, n \geq 1$.

Then for a (smooth) compact Riemann surface, consider the constant sheaf $\underline{\mathbb{Z}}$. If we adopt the analytic topology and view it as a complex manifold, then it can be triangulated, thus the corresponding first-order (Čech) cohomology is $\mathbb{Z}^{2g}$; but if we adopt the Zariski topology, the first-order cohomology vanishes, which provides an example. Next we give a proof of the above theorem in the case of $n = 1$ (there is basically no essential difference for higher-order cases).

Proof. Consider some open cover $\mathfrak{U} = \{U_i\}$ of X (due to the quasi-compactness of X under the Zariski topology, we can assume without loss of generality that the index set is finite, $i = 0, 1, \dots, n$). Let c be some cohomology class in $\check{H}^1(X, \underline{G})$, represented by some 1-cocycle (f_{ij}) . From the definition of the sheaf we know $f_{ii} = 0, f_{ij} = -f_{ji}, f_{ik} = f_{ij} + f_{jk}, \forall i < j < k$ (these elements are all well-defined, because under the Zariski topology any two open sets have an intersection), thus each 1-cocycle (f_{ij}) is in one-to-one correspondence with $(f_{i,i+1})$, because once a set of $(f_{i,i+1})$ is given, we must have $f_{ij} = \sum_{k=i}^{j-1} f_{k,k+1}$, and this set of elements indeed forms a 1-chain. Then let $g_0 = 0, g_i = \sum_{k=0}^{i-1} f_{k,k+1} (i \geq 1)$, then (g_i) is a 0-cochain, and we have $f_{ij} = g_i - g_j$, thus (f_{ij}) is indeed a coboundary, which proves that in $\check{H}^1$ we have $c = 0$, thus $\check{H}^1(X, \underline{G}) = 0$. $\square$

5 Outline of the Proof of the Main Theorem

Next, we briefly describe the proof idea of the main theorem. The basic plan is to first handle the cases on the projective space and the twisted structure sheaf $\mathcal{O}_X(n)$, as these cohomologies are relatively easy to compute "directly", and then the proof for the general case can be given by comparing exact sequences. Specifically, just like the operation when computing the cohomology of an algebraic coherent sheaf, at that time we have a short exact sequence $0 \rightarrow \mathcal{R} \rightarrow \mathcal{E} \xrightarrow{f} \mathcal{F} \rightarrow 0$, where f is a surjection, $\mathcal{R} = \ker f$, and $\mathcal{E}$ is the direct sum of several $\mathcal{O}(n_i)$. Via this short exact sequence, we can then attempt to migrate the situation of $\mathcal{O}(n)$ to the general situation.

For Theorem One, consider a closed immersion $i : X \hookrightarrow \mathbf{P}^r(\mathbb{C})$. We know that a closed immersion is an affine morphism, so by [H] Ex III.4.1. we know $H^k(X, \mathcal{F}) \cong H^k(\mathbf{P}^r(\mathbb{C}), i_* \mathcal{F})$, thus we have directly reduced it to the case on the projective space.

At this point we first consider $\mathcal{F} = \mathcal{O}_X(n)$. To this end, we use induction on the dimension r . The basic tool used is the short exact sequence $0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_E \rightarrow 0$, where E is some hyperplane $t = 0$. Tensoring this exact sequence with $\mathcal{O}(n)$ yields $0 \rightarrow \mathcal{O}(n-1) \rightarrow \mathcal{O}(n) \rightarrow \mathcal{O}_E(n) \rightarrow 0$, thereby through the comparison of the long exact sequences of cohomology respectively induced by $\mathcal{F}$ and $\mathcal{F}^b$ (using the five lemma), we can then reduce to the case of $n = 0$. As for the case of $n = 0$, the global sections in fact only consist of constants, and higher-order cohomologies all vanish (in fact, by Hodge theory, $H^q(X^b, \mathcal{O}^b) \cong H^0(X^b, \Omega_{X^b}^q)$, and by [H] Ex III.7.3. we know that for $q > 0$, $H^0(X^b, \Omega_{X^b}^q) = 0$), which completes the proof of this part.

Next we use the exact sequence $0 \rightarrow \mathcal{R} \rightarrow \mathcal{G} \xrightarrow{f} \mathcal{F} \rightarrow 0$ mentioned previously. Due to the exactness of $(-)^b$, via its action we will directly obtain a corresponding exact sequence in $\mathbf{Coh}(X^b)$. Discussing the cohomologies respectively induced by these two exact sequences using the five lemma can achieve the proof of using the case of $\mathcal{O}(n)$ to complete the general case.

The proof of Theorem Two is actually not difficult. From the perspective of commutative algebra, this theorem essentially says that under the assumption of flatness (and certain finiteness conditions), the Hom functor and the tensor product functor commute. We only need to notice that a sheaf morphism $f : \mathcal{F} \rightarrow \mathcal{G}$ can be regarded as a global section of the sheaf $\mathcal{A} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$; similarly, $g : \mathcal{F}^b \rightarrow \mathcal{G}^b$ can be regarded as a global section of the sheaf $\mathcal{B} = \mathcal{H}om_{\mathcal{H}_X}(\mathcal{F}^b, \mathcal{G}^b)$. In this way, we can apply Theorem One.

Specifically, we can decompose the mapping $f \mapsto f^b : H^0(X, \mathcal{A}) \rightarrow H^0(X^b, \mathcal{B})$ into $H^0(X, \mathcal{A}) \xrightarrow{\varepsilon} H^0(X^b, \mathcal{A}^b) \xrightarrow{i} H^0(X^b, \mathcal{B})$, where ε is exactly the mapping between these two cohomology groups given in Theorem One, and is therefore a bijection. And i is the global section morphism induced by the sheaf morphism $\iota : \mathcal{A}^b = \mathcal{A}' \otimes_{\mathcal{O}_X} \mathcal{H}_X \rightarrow \mathcal{B}$. Locally, ι_x is exactly the canonical homomorphism $\text{Hom}(\mathcal{F}_x, \mathcal{G}_x) \otimes_{\mathcal{O}_x} \mathcal{H}_x \rightarrow \text{Hom}(\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{H}_x, \mathcal{G}_x \otimes_{\mathcal{O}_x} \mathcal{H}_x)$. From conclusions in commutative algebra, since the ring $\mathcal{O}_x$ is Noetherian (or more generally, we only need to assume the finite presentation property of $\mathcal{F}_x$) and $\mathcal{H}_x$ is flat over $\mathcal{O}_x$, this homomorphism is indeed an isomorphism, which proves the second theorem.

Next consider Theorem Three. The uniqueness part in this theorem is not difficult to prove from the exactness of the analytification functor $(-)^b$. Below we only discuss essential surjectivity. To prove the essential surjectivity of this functor, we similarly hope to first reduce it to the case on the projective space. This part essentially only requires the commutativity between the extension-by-zero functor j_* and the analytification functor; reducing to the local level is guaranteed by Theorem 3.2, and specific details are omitted here.

For the case on the projective space, the core of this proof lies in directly obtaining a similar version of a short exact sequence in $\mathbf{Coh}(X^b)$ and making a comparison. In other words, we hope to prove that for every analytic sheaf $\mathcal{M}$, there exists a short exact sequence $0 \rightarrow \mathcal{R} \rightarrow \mathcal{L} \xrightarrow{f} \mathcal{M} \rightarrow 0$, where f is a surjection, $\mathcal{R} = \ker f$, and

$\mathcal{L} = \mathcal{H}(-n)^p = (\mathcal{O}(-n)^p)^b$, where n is some sufficiently large positive integer.

To this end, we first need to define $\mathcal{M}(n)$ for an analytic sheaf $\mathcal{M}$. This construction is completely identical to the construction on a general algebraic sheaf, so it will not be detailed. Then similarly, the existence of the above short exact sequence is simply saying:

Lemma 5.1. For an analytic coherent sheaf $\mathcal{M}$ on $X = \mathbb{P}^r(\mathbb{C})$, there exists a positive integer $n = n(\mathcal{M})$, such that for all $m \geq n$, the $\mathcal{H}_x$ -module $\mathcal{M}(n)_x$ can all be generated by global sections (i.e. $H^0(X^b, \mathcal{M}(n))$).

In addition to this, we also need a lemma similar to the vanishing of higher cohomologies in the algebraic version:

Lemma 5.2. Suppose E is a hyperplane in $X = \mathbb{P}^r(\mathbb{C})$, and $\mathcal{A}$ is an analytic coherent sheaf on E , then for all $q > 0$, there exists a sufficiently large positive integer n such that $H^q(E^b, \mathcal{A}(n)) = 0$.

Next, we use induction on r to prove these two lemmas and Theorem Three (its essential surjectivity) itself. At this point, Lemma 5.2 is obvious, because applying the induction hypothesis to the projective variety E of dimension $r - 1$, by Theorem One we can reduce it to the algebraic case, where this is a classical conclusion.

Then to prove Lemma 5.1, first notice that, similar to the classical situation in algebraic geometry, the compactness of X^b brings a finite open cover and the extension guaranteed by Nakayama's lemma (a basis at a point can be extended to a neighborhood of this point), so we can "weaken" Lemma 5.2 into a version dependent on the selection of point x : for any $x \in X$, there exists a positive integer $n = n(x, \mathcal{M})$, such that $H^0(X^b, \mathcal{M}(n))$ generates $\mathcal{M}(n)_x$.

(Above, we omitted many relatively familiar technical methods and details in the proof, and next we will give the remaining parts of the proofs of Lemma 5.1 and Theorem 4.3 in more detail.)

Then consider a hyperplane $E : t = 0$ passing through x , let $\mathcal{A}$ be the corresponding ideal sheaf of E , thus we have the exact sequence $0 \rightarrow \mathcal{A}(E) \simeq \mathcal{H}(-1) \xrightarrow{t} \mathcal{H} \rightarrow \mathcal{H}_E \rightarrow 0$. Tensoring with $\mathcal{M}$ we get the exact sequence $\mathcal{M} \otimes \mathcal{A}(E) \simeq \mathcal{M}(-1) \rightarrow \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{H}_E \rightarrow 0$. Denote $\mathcal{B} = \mathcal{M} \otimes \mathcal{H}_E$, and denote $\mathcal{C} = \ker \mathcal{M} \otimes \mathcal{A}(E) \rightarrow \mathcal{M}$. (In fact, since $\otimes_{\mathcal{H}} \mathcal{H}$ is naturally an exact functor, we have $\text{Tor}_1(\mathcal{M}, \mathcal{H}) = 0$, thus actually $\mathcal{C} = \text{Tor}_1(\mathcal{M}, \mathcal{H}_E)$.) In short, we then have the exact sequence $0 \rightarrow \mathcal{C} \rightarrow \mathcal{M}(-1) \rightarrow \mathcal{M} \rightarrow \mathcal{B} \rightarrow 0$. Similarly, "twisting" it n times yields the exact sequence $0 \rightarrow \mathcal{C}(n) \rightarrow \mathcal{M}(n-1) \rightarrow \mathcal{M}(n) \rightarrow \mathcal{B}(n) \rightarrow 0$.

Next split this exact sequence into two parts, i.e., let $\mathcal{P}_n = \ker \mathcal{M}(n) \rightarrow \mathcal{B}(n) = \text{im } \mathcal{M}(n-1) \rightarrow \mathcal{M}(n)$, thus the above exact sequence is equivalent to two exact sequences

$$0 \rightarrow \mathcal{C}(n) \rightarrow \mathcal{M}(n-1) \rightarrow \mathcal{P}_n \rightarrow 0$$

$$0 \rightarrow \mathcal{P}_n \rightarrow \mathcal{M}(n) \rightarrow \mathcal{B}(n) \rightarrow 0$$

Then consider (a part of) the long exact sequences of cohomology induced by them:

$$\begin{aligned} H^1(X^b, \mathcal{M}(n-1)) &\rightarrow H^1(X^b, \mathcal{P}_n) \rightarrow H^2(X^b, \mathcal{C}(n)) \\ H^1(X^b, \mathcal{P}_n) &\rightarrow H^1(X^b, \mathcal{M}(n)) \rightarrow H^1(X^b, \mathcal{B}(n)) \end{aligned}$$

Since $\mathcal{C}$ is defined as $\ker \mathcal{M} \otimes \mathcal{A}(E) \rightarrow \mathcal{M}$, it is seen that for all $x \notin E$, $\mathcal{C}_x = 0$, thus $\mathcal{C}$ can actually be viewed as a (coherent analytic) sheaf on $\mathcal{H}_E$, and the same is seen for $\mathcal{B}$. Therefore we can apply Lemma 5.2 to them, seeing that there exists a positive integer n_0 , such that for all $n \geq n_0$, $H^2(X^b, \mathcal{C}(n)) = H^1(X^b, \mathcal{B}(n)) = 0$ holds. Then at this time in the above exact sequence we can see that $\dim H^1(X^b, \mathcal{M}(n-1)) \geq \dim H^1(X^b, \mathcal{P}_n) \geq \dim H^1(X^b, \mathcal{M}(n))$. If we let $\Phi(n) = \dim H^1(X^b, \mathcal{M}(n))$, then $\Phi(n)$ is a monotonically decreasing function taking non-negative integer values, therefore there exists $n_1 \geq n_0$, such that for all $n \geq n_1$, the function $\Phi(n)$ is a constant value. Then $\dim H^1(X^b, \mathcal{M}(n-1)) = \dim H^1(X^b, \mathcal{P}_n) = \dim H^1(X^b, \mathcal{M}(n))$. But since $n_1 \geq n_0$, we still have $H^2(X^b, \mathcal{C}(n)) = H^1(X^b, \mathcal{B}(n)) = 0$, then the two exact sequences above give surjections between vector spaces of the same dimension, which are thus isomorphisms.

We once again return to the long exact sequence of $0 \rightarrow \mathcal{P}_n \rightarrow \mathcal{M}(n) \rightarrow \mathcal{B}(n) \rightarrow 0$, and write out more terms:

$$\begin{aligned} 0 \rightarrow H^0(X^b, \mathcal{P}_n) &\rightarrow H^0(X^b, \mathcal{M}(n)) \xrightarrow{\psi} H^0(X^b, \mathcal{B}(n)) \\ &\xrightarrow{\varphi} H^1(X^b, \mathcal{P}_n) \rightarrow H^1(X^b, \mathcal{M}(n)) \rightarrow H^1(X^b, \mathcal{B}(n)) \end{aligned}$$

The previous discussion tells us that $H^1(X^b, \mathcal{P}_n) \rightarrow H^1(X^b, \mathcal{M}(n))$ is also an injection, thus $\text{im } \varphi = 0$, then $\text{im } \psi = \ker \varphi = H^0(X^b, \mathcal{B}(n))$, i.e. $H^0(X^b, \mathcal{M}(n)) \xrightarrow{\psi} H^0(X^b, \mathcal{B}(n))$ is a surjection.

Then, by the induction hypothesis, $\mathcal{B} = \mathcal{G}^b$ for some algebraic coherent sheaf $\mathcal{G}$ on E . Then by Theorem 4.1 we know $H^0(X^b, \mathcal{B}(n)) = H^0(X, \mathcal{G}(n))$. Then from the conclusion of the algebraic case we know that for a sufficiently large n , $\mathcal{G}(n)$ is generated by global sections, i.e., $H^0(X, \mathcal{G}(n))$ generates $\mathcal{G}(n)_x$. Then corresponding to the analytic case, namely $H^0(X, \mathcal{B}(n))$ generates $\mathcal{B}(n)_x$. Next we prove that the n chosen in this way satisfies the requirements of Lemma 5.2.

In fact, this is purely commutative algebra, because $\mathcal{B}(n)_x = \mathcal{M}(n)_x \otimes \mathcal{H}_{x,E}$. Let $A = \mathcal{H}_x$, $M = \mathcal{M}(n)_x$, $\mathfrak{p} = \mathcal{A}_x(E)$, and let N be the subgroup generated by $H^0(X^b, \mathcal{M}(n))$. Then from the preceding discussion we know $H^0(X^b, \mathcal{M}(n)) \rightarrow H^0(X^b, \mathcal{B}(n)) \rightarrow \mathcal{B}(n)_x$ is surjective, thus the canonical image of N in $\mathcal{B}(n)_x = M \otimes_A A/\mathfrak{p} = M/\mathfrak{p}M$ generates $M/\mathfrak{p}M$, i.e. $M = N + \mathfrak{p}M$. Then consider the maximal ideal of ring A (which naturally contains $\mathfrak{p}$), we naturally have $M = N + \mathfrak{m}M$. By Nakayama's lemma we see $M = N$, which completes the proof of Lemma 5.2.

Finally, we give the proof of Theorem 4.3. For any analytic coherent sheaf $\mathcal{M}$ on $X = \mathbb{P}_r(\mathbb{C})$, Lemma 5.2 is actually saying that there exists a positive integer n such that $\mathcal{M}(n)$ is some quotient sheaf of $\mathcal{H}^{\oplus p}$, thus $\mathcal{M}$ is some quotient sheaf of $\mathcal{H}(-n)^{\oplus p}$. Let $\mathcal{L}_0 = \mathcal{O}(-n)^{\oplus p}$, we then have the exact sequence $0 \rightarrow \mathcal{K} = \ker \psi \rightarrow \mathcal{L}_0^b \rightarrow \mathcal{M} \rightarrow 0$. Repeat the above process for $\mathcal{K}$, i.e. consider a surjective analytic homomorphism $\mathcal{L}_1^b \xrightarrow{g} \mathcal{K}$,

where $\mathcal{L}_1^b$ is also of the form $\mathcal{H}(-m)^{\oplus q}$, then we have an exact sequence $\mathcal{L}_1^b \xrightarrow{g} \mathcal{L}_0^b \rightarrow \mathcal{M} \rightarrow 0$. By Theorem 4.2, there exists $f : \mathcal{L}_1 \rightarrow \mathcal{L}_0$ such that $g = f^b$. Then let $\mathcal{F} = \text{coker } f$, we then have the exact sequence $\mathcal{L}_1 \xrightarrow{f} \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0$. Applying the exact functor $(-)^b$ to it, and using the five lemma to compare this exact sequence with the aforementioned exact sequence, we see $\mathcal{M} \cong \mathcal{F}^b$, which finally proves Theorem 4.3.

6 Several Applications of the Main Theorem

In this section, we give several applications of the main theorem. Among them, Theorems 6.1 and 6.3 both come from the original GAGA text, while another application regarding algebraic fiber bundles in the original text is omitted here.

We use $\Omega_{X/\mathbb{C}}^\bullet$ to denote an algebraic de Rham complex on a complex algebraic variety, that is, $\Omega_{X/\mathbb{C}}^i = \bigwedge^i \Omega_{X/\mathbb{C}}^1$, $\Omega_{X/\mathbb{C}}^0 = \mathcal{O}_X$, and the mapping $d = d_{X/\mathbb{C}} : \Omega_{X/\mathbb{C}}^i \rightarrow \Omega_{X/\mathbb{C}}^{i+1}$. For the sake of conciseness, we omit the specific conditions that the mapping d needs to satisfy. One only needs to know that under the action of the analytification functor $(-)^b$, we get the holomorphic de Rham complex $\Omega_{X^b}^\bullet$, where $\Omega_{X^b}^i$ naturally represents the i -th holomorphic differential forms on X^b .

By the main theorem, we know that for all $p, q \in \mathbb{N}$, $\dim H^q(X^b, \Omega_{X^b}^p) = \dim H^q(X^b, (\Omega_{X/\mathbb{C}}^p)^b) = \dim H^q(X, \Omega_{X/\mathbb{C}}^p)$. Then for a non-singular algebraic variety X , by Dolbeault's theorem we know that $b_n(X) = \sum_{p+q=n} b^{p,q}(X)$, where $b^{p,q}(X) = \dim H^q(X^b, \Omega_{X^b}^p)$. The main theorem implies that the calculation of these Betti numbers can still be completely reduced to the algebraic variety itself to be accomplished.

In particular, for each automorphism σ of the complex number field, considering an algebraic variety X^σ obtained from an algebraic variety X under its action, we can immediately see:

Theorem 6.1. If X is smooth, then the various Betti numbers of X and X^σ are the same.

By a similar method, directly comparing the holomorphic de Rham complex and the algebraic de Rham complex, we can supplement another application of the main theorem, where $\mathbb{H}$ represents the hypercohomology of a sheaf complex, i.e. $\mathbb{H}^k(X, \mathcal{F}^\bullet) = b^k(\Gamma(X, \mathcal{F}^\bullet))$, where b is the cohomology of any complex.

Theorem 6.2 (Grothendieck). If X is a smooth projective algebraic variety over $\mathbb{C}$, X^b is the corresponding complex analytic space, and $\Omega_{X/\mathbb{C}}^\bullet$ is the (algebraic) de Rham complex on it. Then there exists a canonical isomorphism $H_{\text{sing}}^k(X^b, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega_{X/\mathbb{C}}^\bullet)$.

Proof. Since $\Omega_{X^b}^\bullet$ is a resolution of $\underline{\mathbb{C}}$, we have $H^k(X^b, \underline{\mathbb{C}}) \cong \mathbb{H}^k(X^b, \Omega_{X^b}^\bullet)$. On the other hand, we have the following spectral sequences:

$$\begin{aligned} E_{1,b}^{p,q} &= H^q(X^b, \Omega_{X^b}^p) \Rightarrow \mathbb{H}^k(X^b, \Omega_{X^b}^\bullet) \\ E_1^{p,q} &= H^q(X, \Omega_{X/\mathbb{C}}^p) \Rightarrow \mathbb{H}^k(X, \Omega_{X/\mathbb{C}}^\bullet). \end{aligned}$$

By the main theorem, we know there exists a canonical isomorphism $E_{1,b}^{p,q} \cong E_1^{p,q}$, thus there is a canonical isomorphism $\mathbb{H}^k(X^b, \Omega_{X^b}^\bullet) \cong \mathbb{H}^k(X, \Omega_{X/\mathbb{C}}^\bullet)$. On the other hand, we mentioned in Remark 4.1 that the sheaf cohomology of $\underline{\mathbb{C}}$ is the same as the singular homology, therefore $H_{\text{sing}}^k(X^b, \mathbb{C}) \cong H^k(X^b, \underline{\mathbb{C}}) \cong h^k(\Gamma(X^b, \Omega_{X^b}^\bullet)) = \mathbb{H}^k(X^b, \Omega_{X^b}^\bullet) \cong \mathbb{H}^k(X, \Omega_{X/\mathbb{C}}^\bullet)$. $\square$

Finally, we prove Chow's theorem and some of its corollaries.

Theorem 6.3 (Chow). Every closed analytic subset of a projective space is an algebraic variety.

Proof. Let X be a projective space and Y be a closed analytic subset in X^b . By a theorem of Cartan mentioned in the first part, $\mathcal{H}_Y \cong \mathcal{H}_X/\mathcal{A}(Y)$ is a coherent sheaf. Then by Theorem 4.3 we know there exists a coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}^b = \mathcal{H}_Y$. Since $\mathcal{F}^b$ and $\mathcal{F}$ have the same support (because $\mathcal{F}_x^b = \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{H}_x$ and $(\mathcal{O}_x, \mathcal{H}_x)$ constitutes a flat pair), then from the fact that $\mathcal{F}$ is a coherent sheaf we know that $\text{supp } \mathcal{F}^b = \text{supp } \mathcal{F}$ is a Zariski closed set, thus Y is a Zariski closed set, which completes the proof. $\square$

Applying some additional results, we can prove the following corollary. Specific details can be found in the original text:

Corollary 6.1. Compact analytic subsets of algebraic varieties are all algebraic varieties.

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