

Geometric Littlewood-Richardson Rule

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Date: December 29, 2024

Abstract

In this paper, we will go through the paper *A Geometric Littlewood-Richardson Rule* of Vakil. We will first describe this rule in a pure combinatorial way which is just translated from its geometric version. Then we will exhibit the geometric degeneration process with an explicit example and state the main theorem. Finally we will briefly introduce the technical tools used for the proof and give a coarse sketch of the proof.

1 Geometric Settings: Schubert Calculus Revisited

We shall first revise the basic geometric settings that we need to explain the so-called “geometric” Littlewood-Richardson rule, which is the Schubert variety and its cohomology.

Suppose λ is a Young diagram that fits into $(n - k)^k$, and fix for now a (complete) flag:

$$F_{\bullet} = (\{0\} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n = \mathbb{C}^n).$$

We shall denote the *Schubert cell* corresponding to λ to be

$$\Omega_{\lambda}^{\circ}(F_{\bullet}) = \{V \in \text{Gr}(k, n) \mid \dim(V \cap F_j) = i \text{ for} \\ n - k + i - \lambda_i \leq j \leq n - k + i - \lambda_{i+1}\}.$$

while the corresponding *Schubert variety* is

$$\Omega_{\lambda}(F_{\bullet}) = \{V \mid \dim(V \cap F_{n-k+i-\lambda_i}) \geq i, \forall i\}.$$

We shall omit the basic elementary algebraic facts of them here, only stressing that the Schubert cells give a cell-decomposition of the Grassmannian $\text{Gr}(k, n)$, and its closure is just the corresponding Schubert variety. In light of this, we know those Schubert cycles form the basis of the cohomology ring (with coefficients in $\mathbb{Z}$), i.e., $H^*(\text{Gr}(k, n), \mathbb{Z}) = \bigoplus_{\lambda \subseteq (n-k)^k} \mathbb{Z}[\Omega_{\lambda}]$, where $[\Omega_{\lambda}] \in H^{2|\lambda|}(\text{Gr}(k, n), \mathbb{Z})$. To deduce the multiplicative structure, we need also the following fact from intersection theory:

Lemma 1.1. *For two subvarieties V, W , if $V \cap W = \bigcup_{i=1}^n Z_i$, where Z_i are irreducible components, when this intersection is proper and transverse, $[V] \cdot [W] = [Z_1] + [Z_2] + \cdots + [Z_n]$.*

Together with the facts above and the Kleiman-Bertini theorem, one can usually efficiently reduce the computation of the multiplicative structure coefficient of the cohomology ring into a enumerative problem on intersections, via a translate to ensure the intersection is transverse.

An exemplifying computation of the philosophy above is the following

Theorem 1.2 (Duality). *Suppose $|\lambda| + |\mu| = k(n - k)$, then $[\Omega_\lambda] \cdot [\Omega_\mu] = \delta_{\lambda, \mu^c} [\Omega_{(n-k)^k}]$*

Moreover, one can prove the geometric Pieri's rule via the same way, using the duality above, and compute the intersection with two opposite flags and another generic one. The resulting rule is exactly the same as that in the context of symmetric polynomial. In fact, this determines the whole structure of the cohomology ring. Compared with the exactly same rule in the context of symmetric polynomials, we have

Theorem 1.3. $H^*(Gr(k, n)) \cong \Lambda / \langle s_\lambda | \lambda \not\subseteq (n - k)^k \rangle$ via $[\Omega_\lambda] \mapsto s_\lambda$.

Even so, one still wants to know if there is a direct way to depict (a combinatorial model of) the general multiplicative coefficient $C_{\lambda\mu}^\nu$, where $[\Omega_\lambda] \cdot [\Omega_\mu] = \sum_\nu C_{\lambda\mu}^\nu [\Omega_\nu]$.

This is done in this paper, but with a different method as above: not by taking “triple” transversal intersection, but through a degeneration process, a method which the author dates back to Pieri himself.

2 Combinatorial Description via Checker Game

In fact, the final geometric result can be directly translated into a combinatorial version in the guise of a “checker game”, hence before discussing the geometric process, we shall first introduce this more intuitive and easy-to-describe version in this section.

First we shall introduce the notations. Consider an $n \times n$ board, whose rows and columns are numbered as in Figure 1. We can fill those grids with some *checkers*. In particular, no two checkers of are in the same row or column, thus it can be regarded as a permutation over $[n]$. However, the bijection here is somehow “twisted”, i.e., we shall read it from right to left, i.e. the permutation sends $n + 1 - c$ to r . The permutation in Figure 1 is an example.

Also, in the following paper, we will usually use the bijection between a Young diagram fitting in $(n - k)^k$, with a k -element subset of $[n]$. The bijection is well-known and is shown in Figure 2.

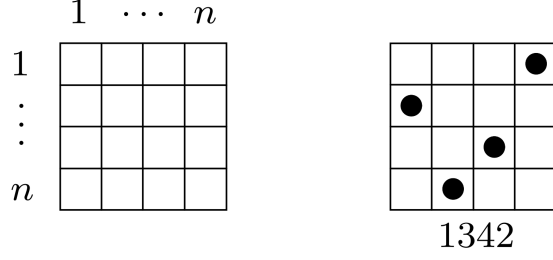


Figure 1: Bijection of a permutation with the checkers

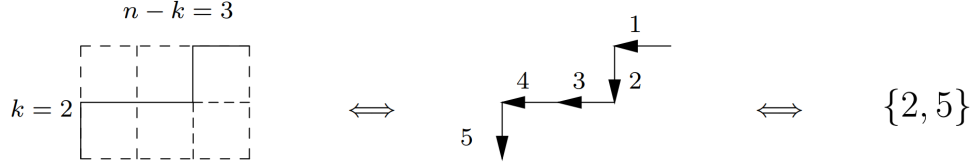


Figure 2: Bijection of a Young diagram with a subset

Under the bijection above, we shall consider a checker game consisting of n black checkers moving from “1” to “ w_0 ”, the longest element in $\mathfrak{S}_n$, and k white checkers, whose initial position corresponds to two k element subsets $A, B \subseteq [n]$ (in bijection with partitions α, β). Through a series of “legal” movements, the final possible position of the white checkers corresponds to one k element subset C of $[n]$ (in bijection with a partition γ), then the main theorem can be expressed as follows:

Theorem 2.1. *Given α, β as input of a checker game, then $\sum_{\text{games } G} s_{\text{output } G} = \sum_{\gamma} C_{\alpha\beta}^{\gamma} s_{\gamma}$, where $C_{\alpha\beta}^{\gamma}$ is the desired Littlewood-Richardson coefficients.*

Next, we shall first introduce how the black checkers move, and then define the initial position and what is a “legal” move for the white checkers.

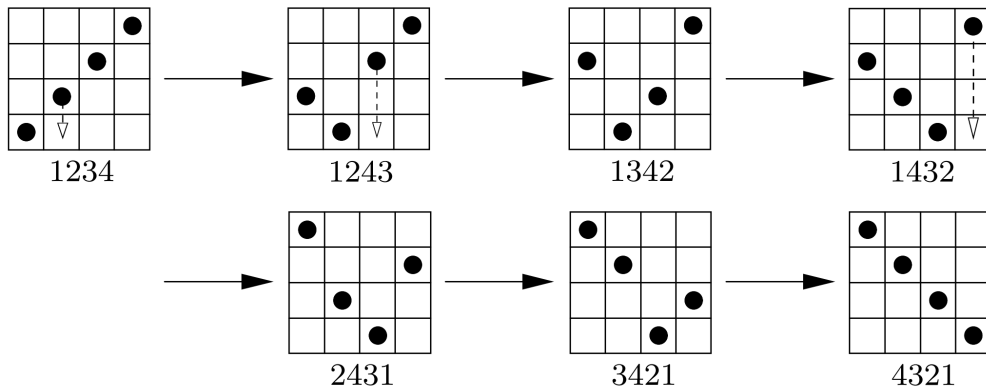


Figure 3: Move of the black checkers

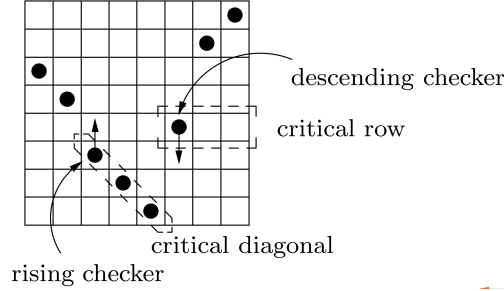


Figure 4: Critical row and diagonal

The move of the black checkers are just in fact the same process as the bubble-sort from the identity to the longest element w_0 in $\mathfrak{S}_n$, via a “twisted correpondence” to the checkerboard, just subjecting to a certain “specialization” order, as shown in Figure 3. To write in a formal way, consider the reduced expression of $w_0 = s_{n-1}s_{n-2} \cdots s_1s_{n-2}s_{n-3} \cdots s_1 \cdots s_{n-1}s_{n-2}s_{n-1}$, the “specialization” order corresponds to its suffix, i.e., $s_{n-1}, s_{n-2}s_{n-1}, s_{n-1}s_{n-2}s_{n-1}, \cdots$.

Note that during this process, there are always two black checkers in the adjacent rows, one descending, one rising. We shall define the *critical row* to be the row which the descending checker is in. And note that by the description above of the process, the checkers below the descending checker are always in a diagonal position. We shall define this diagonal to be the *critical diagonal*.

For white checkers, we shall first define what means to be “happy”. Generally, we shall call a white checker happy if there are both black checkers above it or to its left (including the case when it is in the same square with some black checker). A configuration of white checkers is happy when every white checker in it is happy. For example, if any white checker is in the same square with some black checker, then the whole configuration is happy. Also, if the black checkers are on the main diagonal, this is the only possiblity for white checkers to be happy.

The initial position of the checkers are given by two sets $A = \{a_1, \cdots, a_k\}, B = \{b_1, \cdots, b_k\}$ (recall the bijection of them with the Young diagrams), with $a_1 < \cdots < a_k, b_1 < \cdots < b_k$. The initial position of the white checkers are then $(a_1, b_k), (a_2, b_{k-1}), \cdots, (a_k, b_1)$.

Then we shall discuss the move of the white checkers, including two phases. The first phase is intrinsic while the second phase is just “happification”.

To be concise, in the first phase, the only two white checkers that we are deciding whether to move or not are the one in the critical row and the top one in the critical diagonal (if there is any in both case). If either one is in the same square as the moving black checker, then swap them. The only possible two-branching case when they can either swap or stay unchanged happen if it is not the case above, and there is no “blocker” between them. (The action in the first phase is summarized in the following table in Figure 5, which I think is seemingly too complicated.)

		White checker in critical row?		
		yes, in descending checker's square	yes, elsewhere	no
Top white checker in critical diagonal?	yes, in rising checker's square	swap	swap	stay
	yes, elsewhere	swap	swap if no blocker or stay	stay
	no	stay	stay	stay

Figure 5: Deciding whether swap or not

Notice that after the first phase, a white checker might only have a black checker to its left or above it, then in the second phase, we need to make it happy, i.e., move it straightforward to the same square as that black checker.

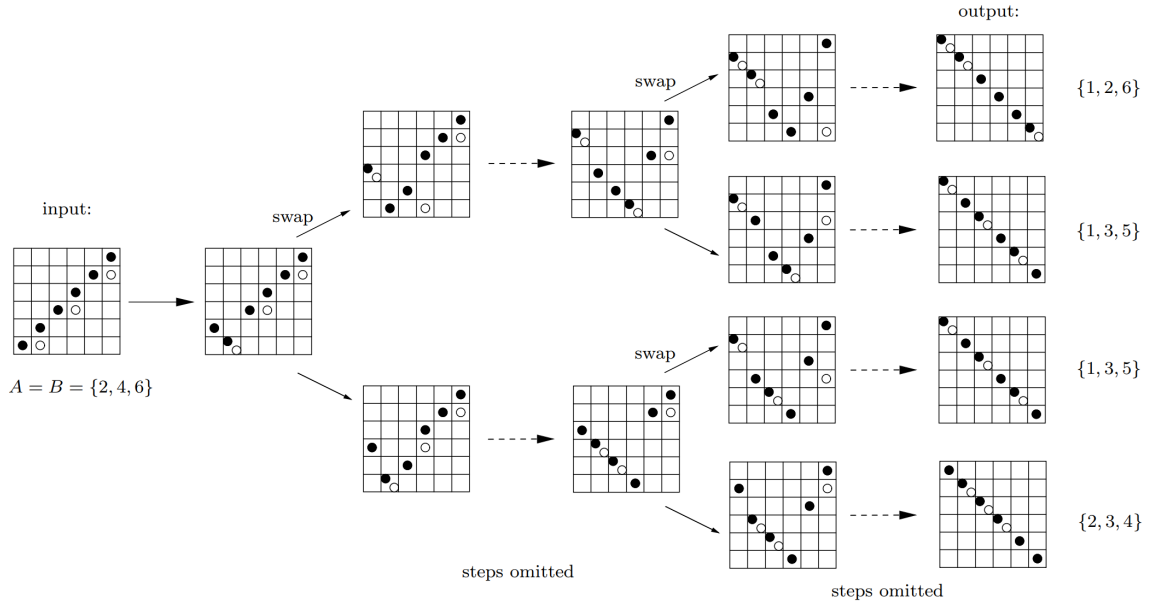


Figure 6: Calculation example

The process in Figure 6 is an example calculating $C_{(2,1),(2,1)}^{(3,2,1)} = 2$ when $(k, n) = (3, 6)$ (and also give other coefficients, though there is a typo on the index), using the Littlewood-Richardson rule in this version.

3 The Geometric Process: Degeneration

We shall describe the geometric process in this section. The general strategy is to deform the moving flag $M_\bullet$ step by step until it coincides with the fixed flag $F_\bullet$. We care about the variety $\Omega_A(M_\bullet) \cap \Omega_B(F_\bullet)$ after the process (where A, B are two k -element sets in bijection with two Young diagrams λ, μ). Namely, after the deformation, the Schubert cycle that $\Omega_A(M_\bullet) \cap \Omega_B(F_\bullet)$ corresponds to, is the desired intersection product of $[\Omega_A] \cdot [\Omega_B]$, while now the flag in consideration is certainly $F_\bullet$, hence we only need to find out what the final variety is, which will give us the geometric Littlewood-Richardson rule.

Hence this section will be presented in two parts. In the former part we shall describe the initial variety $\Omega_A(M_\bullet) \cap \Omega_B(F_\bullet)$ as a certain two-flag variety, and correspond it with the checker game we stated before by comparing the checkers' configuration and the varieties' geometric landscape. The latter part consists of the explicit deformation process (and also the checker moving process), where we describe how the initial variety changes in this process and the deforming result.

3.1 Definition and Correspondence

We shall first focus on the relative positions of the two flags $M_\bullet$ and $F_\bullet$. Recall that any configuration of black checkers is in bijection a permutation on $[n]$, and one can regard it as a permutation on n vectors $p_1, p_2, \dots, p_n \in \mathbb{C}^n$ in general positions. Here we suppose $F_\bullet = (\{p_n\} \subset \{p_n, p_{n-1}\} \subset \dots \subset \mathbb{C}^n)$ and the moving flag is its opposite flag initially.

Consider a configuration $\bullet$ of our game board with only **black checkers**. From each configuration $\bullet$ to its next move $\bullet_{\text{next}}$, we exchange exactly two vectors in the series of $M_\bullet$. Then by permuting vectors in the specialization order we will finally deform $M_\bullet$ into $F_\bullet$.

One can calculate the dimension from this process. We say a square (i_1, j_1) dominates another square (i_2, j_2) if $i_1 \geq i_2$ and $j_1 \geq j_2$. We have a direct enumeration that

Proposition 3.1. *Suppose the relative position of $M_\bullet$ and $F_\bullet$ is given by $\bullet$. Then*

$$\dim(M_i \cap F_j) = \#\{(i', j') \mid \bullet \text{ in } (i', j'), (i, j) \text{ dominates } (i', j')\}$$

Thus we get a bijection between configurations $\bullet$ and relative positions of those two flags. Note that although we demand that $\bullet$ fits certain step among the specialization order, those that don't meet the requirements can also lead to a pair of flags. Please refer to the example in Figure 7. The one above is in the specialization order while the one below is not.

We write the variety $X_\bullet$ to be the subvariety of $Fl(n) \times Fl(n)$ corresponding to flags $(M_\bullet, F_\bullet)$ in relative position given by black checker configuration $\bullet$. By definition, $X_\bullet$ should be a locally closed, smooth subvariety of

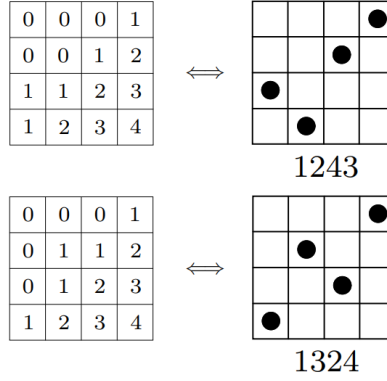


Figure 7: Examples of bijection between $\bullet$ and $(M_\bullet, F_\bullet)$

$Fl(n) \times Fl(n)$, whose codimension is the number of pairs of black checkers in $\bullet$ that one dominates the other. The fact about codimension can be explicitly checked by induction since we have the specialization order, one can also imagine that each pair of black checkers in this case produces an “intersection” which reduces the dimension by 1.

This observation can be vital. Because if we demand our process of deforming reduce one dimension per step, it turns out the specialization order is the only valid sequence of operators that fulfills the requirement.

Next we are going to adding **white checkers**. Suppose $(M_\bullet, F_\bullet) \in X_\bullet$, i.e. these two flags are in relative position given by $\bullet$. For an arbitrary k -subspace $V \in Gr(k, n)$, we have a parameterization based on $(M_\bullet, F_\bullet)$, namely $\dim(V \cap M_i \cap F_j)$. If we take $GL(n)$ -actions that fix two flags into consideration, all $\dim(V \cap M_i \cap F_j)$'s determines the position of V relative to $(M_\bullet, F_\bullet)$.

Since we use $\{p_1, \dots, p_n\} \subset \mathbb{C}^n$ to present the two flags, not caring where they are, that k -subspace V can be presented by them, too; which is described by white checkers. We claim that there exists a bijection given as follows: given a checker configuration $\circ\bullet$, datum $\dim(V \cap M_i \cap F_j)$ is the number of white checkers in squares dominated by square (i, j) ; and given all data of dimension and a black checker configuration $\bullet$, we get a $\circ\bullet$ vice versa.

Hence if we already have $(M_\bullet, F_\bullet)$ fitting $\bullet$, all k -subspace V fitting $\circ\bullet$ relative to $(M_\bullet, F_\bullet)$ consist an open cell in the Grassmannian. Denote that by *open two-flag Schubert variety* $Y_{\circ\bullet} \subset Gr(k, n)$. And the *closed two-flag Schubert variety* $\overline{Y_{\circ\bullet}}$ is defined to be the clousure $Cl_{Gr(k, n)} Y_{\circ\bullet}$. Now take all posible position of $(M_\bullet, F_\bullet)$ fitting $\bullet$ into consideration, we get the *univarsal open (resp. closed) two-flag Schubert variety* in $Gr(k, n) \times X_\bullet$.

The following proposition tells about the dimension of the variety $Y_{\circ\bullet}$, which is proved by similar linear algebraic operations hence we will omit the proof, but let's point out that this is indeed a basic tool to start with.

Proposition 3.2. *The variety $Y_{\circ\bullet}$ is irreducible and smooth; its dimension is the sum over all white checkers w of the number of black checkers that w dominates, minus the number of white checkers that w dominates.*

With the proposition above, the dimension of $\bar{Y}_{\circ, A, B \bullet_{\text{init}}}$ is $\sum_{a \in A} a + \sum_{b \in B} b - k(n+1)$, while that of $\Omega_A(M_\bullet) \cap \Omega_B(F_\bullet)$ is $(\sum_{a \in A} a - \frac{k(k+1)}{2}) + (\sum_{b \in B} b - \frac{k(k+1)}{2}) - k(n-k) = \sum_{a \in A} a + \sum_{b \in B} b - k(n+1)$. We can prove that the initial variety is indeed given by the two-flag variety $\bar{Y}_{\circ, A, B \bullet_{\text{init}}}$ with this fact and further discussion.

Proposition 3.3. *Suppose $A = \{a_1, \dots, a_k\}, B = \{b_1, \dots, b_k\}, M_\bullet, F_\bullet$ are two transverse (opposite) flags with relative position given by $\bullet_{\text{init}}$. Then $\Omega_A(M_\bullet) \cap \Omega_B(F_\bullet)$ is the closed two-flag variety $\bar{Y}_{\circ, A, B \bullet_{\text{init}}}$.*

This settles the characterization of the initial variety.

3.2 Degeneration process

From now on, we are going to give the specific deforming process and compare it with the former checker game.

Following in Figure 8 is an example of $X_\bullet \subset Fl(4) \times Fl(4)$, and we draw it into $\mathbb{P}^3 = \mathbb{C}P^3$, where p_1, p_2, p_3, p_4 are stated as points. In the figures below, we denote $F_\bullet$ by solid lines and a solid point, and $M_\bullet$ by dashed lines and a cross point. One can easily check the equalities about $\dim(M_i \cap F_j)$.

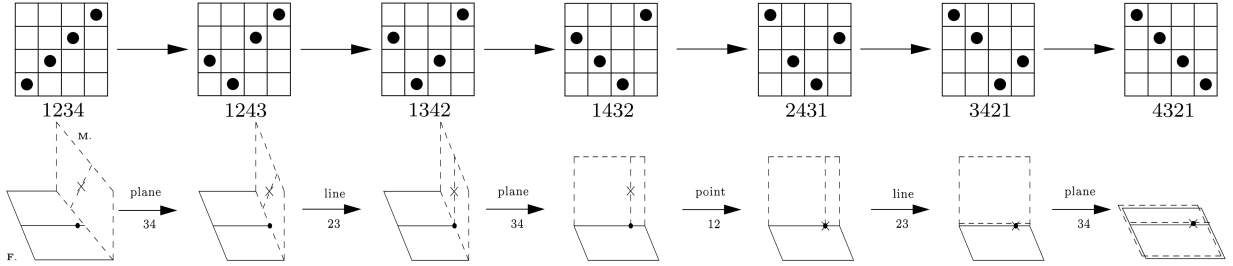


Figure 8: Black checkers moving and two-flag degeneration

Each step above can be regarded as the motion $M_\bullet$ getting close to $F_\bullet$. More precisely, if $\bullet$ is one black checker configuration among the specialization order and $\bullet_{\text{next}}$ is its successor, there is a $\mathbb{P}^1$ -fibration

$$\begin{array}{ccc} X_\bullet \cup X_{\bullet_{\text{next}}} & \longrightarrow & X_{\bullet_{\text{next}}} \\ (p_i \in M_i, p_j \in F_j) & & (p_i \in (M_i \cap F_j)) \end{array}$$

where $\bullet_{\text{next}}$ exchange $i, j \in [n]$ in bijection with permutation.

Next we will discuss how $Y_\bullet$ deforms during $M_\bullet$ degenerates into $F_\bullet$. The process showed in Figure 9 is the same example with Figure 8, but with white checkers and a 2-subspace (viewed as a projective line in $\mathbb{P}^3$) added. We draw the reader's attention to the second step where we fork the process into two routines. Towards the upper one, we make a "swap" since there is no blocker in the checker board; and towards the lower one we simply keep the white checkers stayed. You can refer to Figure 5 if feeling strange to these operations.

In the former case our 2-subspace is stuck to a single point (actually a vector in $\mathbb{C}^4$) (and we can view this as two determining points/vectors of this space also becomes the same), which is the property that *swap* brings.

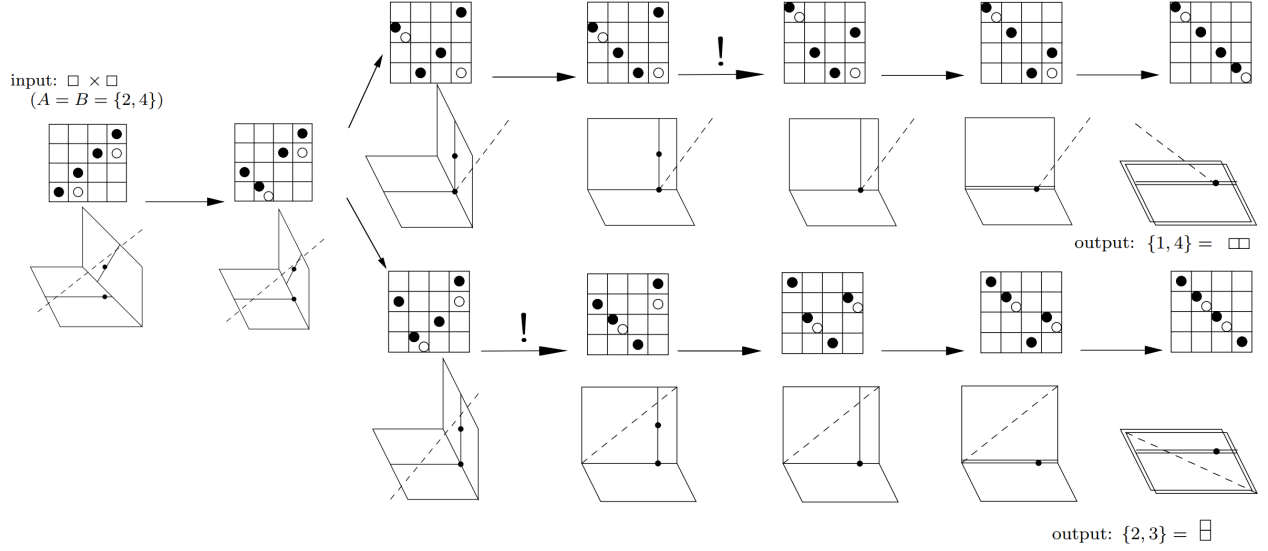


Figure 9: All checkers moving, breaking and further degeneration

The other case shows a more trivial position that 2-subspace lying on. Due to the similarity between Schur polynomials and Schubert cycles, here we just proved $s_{(1)}^2 = s_{(1,1)} + s_{(2)}$.

We are now ready to give a rigorous description of the geometric Littlewood-Richardson rule. We've seen the degeneration base is a $\mathbb{P}^1$, where $\mathbb{A}^1 \in X_\bullet$ and $\{\infty\} \in X_{\bullet, \text{next}}$. Consider a given starting flag, then in fact we have a $\overline{Y}_{\bullet, \bullet}$ -fibration over $\mathbb{A}^1 : \mathcal{Y}_{\bullet, \bullet} \rightarrow \mathbb{A}^1$. Consider the corresponding closure of this fibration in $\text{Gr}(k, n) \times \mathbb{P}^1$, together with another fibre product, we shall get a divisor D_Y that is precisely the variety breaks into during the degeneration process.

$$\begin{array}{ccccc}
 \mathcal{Y}_{\bullet, \bullet} & \hookrightarrow & \text{Cl}_{G(k, n) \times \mathbb{P}^1} \mathcal{Y}_{\bullet, \bullet} & \longleftarrow & D_Y \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 & \longleftarrow & \{\infty\}.
 \end{array}$$

Figure 10: Construction process

The further statement and proof is changed back to the base $X_\bullet \cup X_{\bullet, \text{next}}$, hence we have the following diagram and we care about this D_X .

Then we have main theorem:

Theorem 3.4 (Geometric Littlewood-Richardson Rule). $D_X = \overline{X}_{\bullet, \text{stay} \bullet \text{next}}, \overline{X}_{\bullet, \text{swap} \bullet \text{next}}$ or $\overline{X}_{\bullet, \text{stay} \bullet \text{next}} \cup \overline{X}_{\bullet, \text{swap} \bullet \text{next}}$.

$$\begin{array}{ccccc}
\overline{X}_{\bullet\bullet} := \mathrm{Cl}_{G(k,n) \times X_{\bullet}} X_{\bullet\bullet} & \hookrightarrow & \mathrm{Cl}_{G(k,n) \times (X_{\bullet} \cup X_{\bullet\mathrm{next}})} X_{\bullet\bullet} & \xleftarrow{\quad} & D_X \\
\downarrow & & \downarrow & & \downarrow \\
X_{\bullet} & \hookrightarrow & X_{\bullet} \cup X_{\bullet\mathrm{next}} & \xleftarrow{\quad} & X_{\bullet\mathrm{next}}.
\end{array}$$

Figure 11: Base change back again

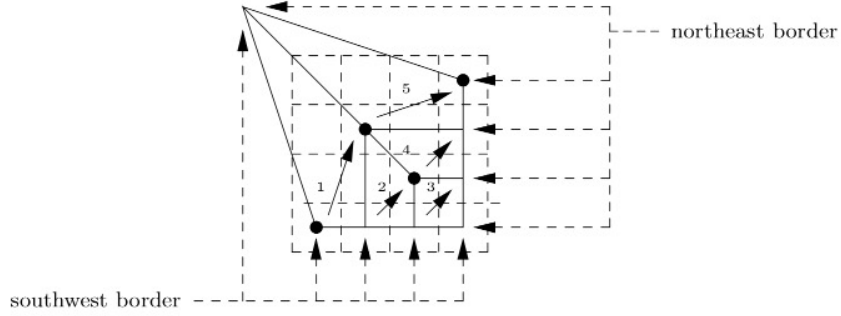


Figure 12: “Bundle” Structure of BS variety

4 Bott-Samelson Variety

In this section we will present a basic technical tool used in the proof, the Bott-Samelson variety. First we shall define a *planar poset*. The element of it is a finite subset of points in the plane, with partial order given by domination (as in the previous sections). It is also equipped with a dim map: $\mathcal{P} \rightarrow \{0, 1, \dots, n\}$, which is weakly order-preserving map. It satisfies that if $\mathbf{b}$ covers $\mathbf{a}$ in $\mathcal{P}$, then $\dim \mathbf{a} = \dim \mathbf{b} - 1$, and also $\mathcal{P}$ has a maximum and a minimum. Then we shall define the corresponding Bott-Samelson variety $BS(\mathcal{P})$ to be the variety parametrizing a dim $\mathbf{s}$ —plane $V_{\mathbf{s}} \in K^n$, with $V_{\mathbf{s}} \subset V_{\mathbf{t}}$ if $\mathbf{s} \leq \mathbf{t}$.

It is somehow an abstract definition. To explain, it can be seen as a generalization of flag varieties. In fact, if the planar poset $\mathcal{P}$ is taken to be an n —element horizontal chain, then the corresponding Bott-Samelson variety is just the flag variety $Fl(n)$. It is also smooth and can be seen as a kind of desingularization of the flag variety.

Another way to intuitively understand BS variety is to view it as inductively adding the data of elements of $s \in S$ corresponding to “new” vertices of quadrilaterals, where the other three vertices are $s_1 \leq s_2 \leq s_3$ are parametrized, and s is the diagonal vertex to s_2 . In this way, it can be viewed as a tower of $\mathbb{P}^1$ —bundles, as in the example in Figure 12.

We can look at the strata structure of BS varieties. In fact, any subset Q of the quadrilaterals in a planar poset determines a stratum of the Bott-Samelson variety. The closed *stratum* corresponds to requiring the subspaces of the opposite corners of the quadrilaterals in Q of the same dimension to be the same. The open stratum corresponds

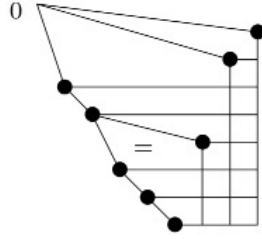


Figure 13: BS variety generated by checkers

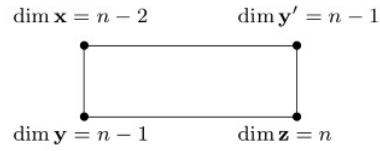


FIGURE 18. The planar poset $\mathcal{P}_{\square}$

Figure 14: Box BS variety

to also requiring the spaces of the opposite corners of the quadrilaterals *not* in Q to be distinct. From the viewpoint above, one can show that the open strata give a stratification; and the closed strata are smooth, and the codimension of the stratum is the size of the subset Q .

We will exhibit next the only two concrete case that will be used in the following paper. The first is the Bott-Samelson variety generated by a black checker configuration, whose elements are the points with simultaneously a black checker above it and to its left, i.e., those “happy” positions (which is defined in section 2), and a minimum $\hat{0}$ added, as shown in the figure 13.

The second is the most simple Bott-Samelson variety $\mathcal{P}_{\square}$, consisting only of four points, as shown in Figure 14. Also notice that this variety has one divisorial stratum, given by $V_y = V_{y'}$, which will be crucial in the proof and will be denoted as $D_{\square}$.

The only remaining definition we need is the BS variety corresponding to a poset morphism. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are planar posets, and $p : \mathcal{P} \rightarrow \mathcal{Q}$ is a morphism of posets. Then let $BS(p) = BS(\mathcal{P} \rightarrow \mathcal{Q})$ be the subvariety of $BS(\mathcal{P}) \times BS(\mathcal{Q})$ such that if $p(s) = t$, then the subspace corresponding to s is contained in the subspace corresponding to t . For example, a configuration of black and white checkers $\bullet \circ$ induces a morphism of posets $p_{\bullet \circ} : \mathcal{P}_{\bullet \circ} \rightarrow \mathcal{P}_{\bullet}$, and $X_{\bullet \circ}$ is an open subset of $BS(p_{\bullet \circ} : \mathcal{P}_{\bullet \circ} \rightarrow \mathcal{P}_{\bullet})$. As another example, Figure 15 depicts

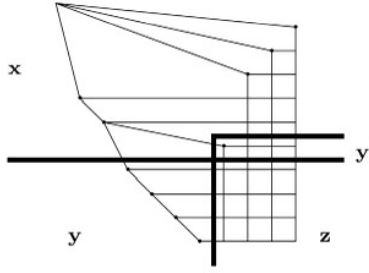


Figure 15: Pictorial Depiction

$p_{\square} : \mathcal{P}_{\bullet} \rightarrow \mathcal{P}_{\square}$ and it is just a kind of “dividing regions”.

Similarly we have $p_{\circ\square}$, and $X_{\bullet\square}$, $X_{\circ\square}$ is defined as a certain open set in the corresponding BS variety of morphism. However, since we will only give the proof sketch, we omit the long and complex definition here for simplicity.

5 Strategy and Sketch of the Proof

We will give out only the total strategy and the main theorems that constitute the whole proof, but still omitting the details. The main strategy is to pull back the most simple divisor $D_{\square}$ mentioned above, and decide what pullbacks are possible by counting the dimension.

The divisors that we want to look at are in the variety $\text{Cl}_{\text{Gr}(k,n) \times (X_{\bullet} \cup X_{\bullet_{\text{next}}})} X_{\bullet\bullet}$. We’ve seen in the example before the geometric picture corresponding to “swap” operation is roughly that two subvarieties in $X_{\bullet\bullet}$ degenerate to be the same, which is the kind of condition we required to define a stratum in $BS(\mathcal{P}_{\circ})$, hence we will first transmit our sight to $\text{Cl}_{BS(\mathcal{P}_{\circ}) \times (X_{\bullet} \cup X_{\bullet_{\text{next}}})} X_{\bullet\bullet}$. Then we will deal with most of the problems first in the language of the Bott-Samelson varieties.

To do this, we need a “triple” fiber product $X_{\bullet\bullet\square}$ obtained as in figure 16.

We will not explain its definition details to avoid heavy notations, but only to admit that this construction allows us to transfer the divisors in $Z := \text{Cl}_{BS(\mathcal{P}_{\circ}) \times BS(\mathcal{P}_{\square})} X_{\bullet\square}$ to the variety in consideration now, i.e., $\text{Cl}_{BS(\mathcal{P}_{\circ}) \times (X_{\bullet} \cup X_{\bullet_{\text{next}}})} X_{\bullet\bullet}$ since we have the following diagram where the smoothness and surjectivity is not hard to prove. Also, Z is a closed subvariety in $BS(\mathcal{P}_{\circ} \rightarrow \mathcal{P}_{\square})$, hence we could reduce the problem to deducing the components of divisors in $BS(\mathcal{P}_{\circ} \rightarrow \mathcal{P}_{\square})$, which finally leads to the easiest divisor $D_{\square}$ in the variety $\mathcal{P}_{\square}$. The whole process is summarized in Figure 17.

So now we need to find the components of the pullback of $D_{\square}$ in $BS(\mathcal{P}_{\circ} \rightarrow \mathcal{P}_{\square})$ which has dimension of at least $\dim X_{\bullet\square} - 1$. To identify them, we need to first define the “good quadrilaterals”.

A *good quadrilateral* in $BS(\mathcal{P}_{\circ})$ is a quadrilateral whose minimal element does not dominate $\mathbf{a}$, and either

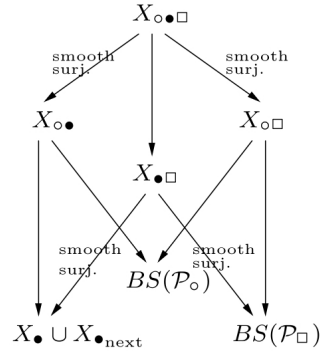


Figure 16: Definition as triple fiber product

$$\begin{array}{ccccc}
 \mathrm{Cl}_{BS(\mathcal{P}_{\circ}) \times (X_{\bullet} \cup X_{\bullet\text{next}})} X_{\circ\bullet} & \xleftarrow{\text{sm. surj.}} & \mathrm{Cl}_{BS(\mathcal{P}_{\circ}) \times (X_{\bullet} \cup X_{\bullet\text{next}})} X_{\circ\bullet\square} & \xrightarrow{\text{sm. surj.}} & Z := \mathrm{Cl}_{BS(\mathcal{P}_{\circ}) \times BS(\mathcal{P}_{\square})} X_{\circ\square} \\
 \downarrow \text{proper birat'l} & & & & \downarrow \text{closed imm.} \\
 \mathrm{Cl}_{G(k,n) \times (X_{\bullet} \cup X_{\bullet\text{next}})} X_{\circ\bullet} & & & & BS(\mathcal{P}_{\circ} \rightarrow \mathcal{P}_{\square})
 \end{array}$$

Figure 17: Pullback of the divisors

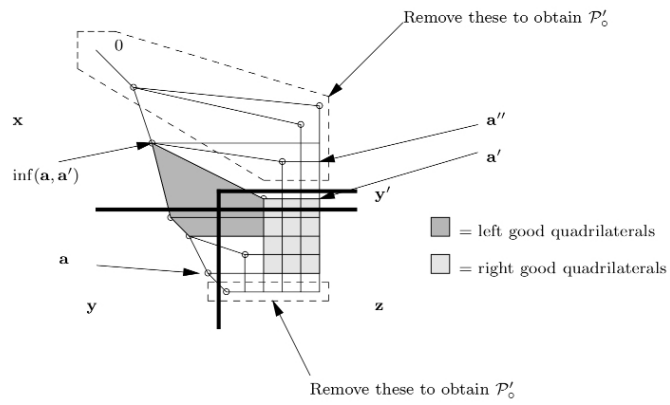


Figure 18: Good quadrilaterals

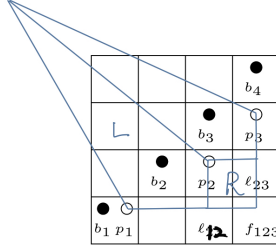


Figure 19: Depiction by checkers

(1) whose minimal elements dominates the white checker $\min(p_{\square}^{-1}(y'))$, or (2) with two vertices dominating $\min(p_{\square}^{-1}(y'))$ and two vertices in $p_{\square}^{-1}(x, y)$. The quadrilateral in the first case will be called a “right” good quadrilateral, while that in the second case will be called a “left” good quadrilateral. Examples are given in Figure 18.

We’ve discussed before that the strata in the Bott-Samelson varieties correspond to adding some requirements to the quadrilaterals. Hence, intuitively, we could just imagine the good quadrilaterals to be those “big enough” strata whose pullback will have enough degree ($\geq \dim X_{\square} - 1$) to serve as a divisor. This is done in the following theorem.

Theorem 5.1. *The components of the pullback of $D_{\square}$ to $BS(\mathcal{P}_{\bullet} \rightarrow \mathcal{P}_{\square})$ of dimension at least $\dim X_{\square} - 1$ correspond exactly to the set S of some good quadrilaterals (with at most one quadrilateral in each column) in the following way. For such a set S , the closure of the open stratum in $BS(\mathcal{P}_{\bullet})$ corresponding to the set S is a component whose dimension is exactly $\dim X_{\square} - 1$.*

It can be believed that the proof of the theorem above is done by heavy discussions but is no more than down-to-earth dimension counting in $BS(\mathcal{P}_{\bullet})$, and also the language of good quadrilaterals is just a method of describing such strata.

Hence the next task is to deduce what set of such quadrilaterals is possible to appear as a divisor at “ ∞ ” in $\text{Cl}_{\text{Gr}(k,n) \times (X_{\bullet} \cup X_{\bullet_{\text{next}}})} X_{\bullet_{\bullet}}$, which is of our original interest. It turns out the only possibility is that $S = \{\}$ or $S = \{\text{top left good quadrilateral}\}$. In fact, we only need to consider the dimension of D_S which is the corresponding divisor to S in the manner of the theorem above. It is called *geometrically irrelevant* if its image in $\text{Gr}(k, n) \times X_{\bullet_{\text{next}}}$ has smaller dimension, and is called *geometrically relevant* otherwise. So in fact we have the following theorem:

Theorem 5.2. *D_S is geometrically relevant if and only if $S = \{\}$ or $S = \{\text{top left good quadrilateral}\}$.*

We will again explain the results above by a concrete example in $\text{Gr}(3, 4)$.

The initial variety is given by the configuration in Figure 19, also we’ve drawn the three quadrilaterals in it, with the left good one labeled “L”, right good one labeled “R”, and the other one is not good.

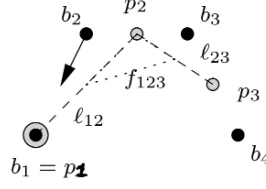


Figure 20: Initial positions

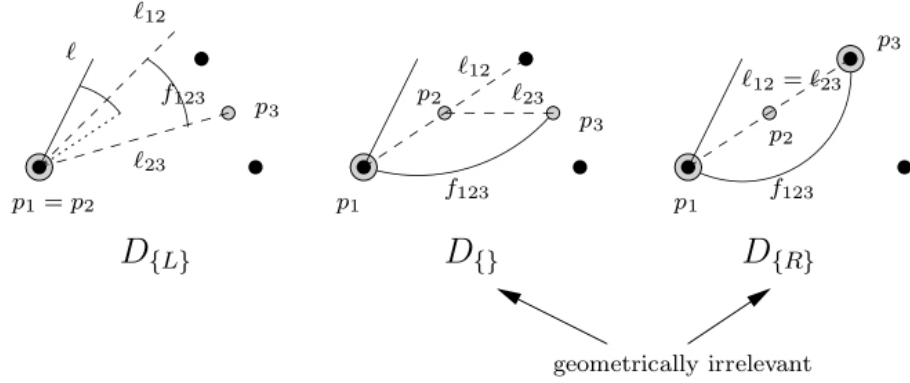


Figure 21: Degeneration cases

So the initial positions of $X_{\bullet\bullet}$ is shown as in the figure 20, in which the subspace F_i is generated by $b_1, \dots, b_i$. Similarly p_i are generating vectors of the variety $X_{\bullet\bullet}$. Then the first degeneration step is to let b_2 approach b_1 in a direction line l , as shown in Figure 20.

We could examine the following possibilities of degeneration of the variety $Y_{\bullet\bullet}$ as in Figure 21. Recall the requirement for a stratum corresponding to a set of quadrilaterals. So the first case corresponds to $S = \{L\}$, i.e., $p_1 = p_2, l_{12} \neq l_{23}, p_2 \neq p_3$. This is geometrically relevant, since for a fixed point in $X_{\bullet\text{next}}$, this locus has dimension 2. Namely, 1 for the choice of l_{12} , which goes through $p_1 = p_2 = b_1 = b_2$, 1 for the choice of p_3 on line $b_3 b_4$. Also, its image in $\text{Gr}(3, 4)$ is two, since the plane f_{123} is determined by l_{12} and p_3 .

However, in the other two cases, the components are not geometrically relevant. This is because in the second case, l_{12} is determined automatically by p_1, p_2 , hence its dimension in $\text{Gr}(3, 4)$ is only 1, depending on the choice of p_3 . The third case is similar. Recalling the geometric example we discussed in $\text{Gr}(2, 4)$, during the degeneration step, we have abandoned such lower-dimension cases at the outset so they are not included as possibilities.

With further consideration, we can show that the above divisors appear with multiplicity 1, and $D_{\emptyset}$ is birational to $\overline{X}_{\circ_{\text{stay}} \bullet_{\text{next}}}$ while $D_{\{\text{top left good quadrilateral}\}}$ is birational to $\overline{X}_{\circ_{\text{swap}} \bullet_{\text{next}}}$, which settles the whole problem.

6 Further Problems

Further problems usually focus on how to generalize this Littlewood-Richardson rule to similar geometric settings, such as for Flag manifolds or symplectic Grassmannians. There are also versions in K-theory and unfinished problems. Besides, we may ask if we could use similar degeneration methods to solve problems in enumerative geometry of surfaces, from the discussion for the quantum cohomology of Grassmannian.