

An application of Hilbert 90 for K_2

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§ 0. INTRODUCTION

In this survey, I will go through the paper [1] of Colliot-Thélène, whose main goal is to prove the following result about the group $A_0(X)$ of classes of degree zero 0-cycles on a rational surface X over a field F of arithmetic nature:

Thm 0.1. Let X be a rational surface over field F , which we suppose to be characteristic 0 for simplicity.

- (a) If F is finitely generated over $\mathbb{Q}$, or a local field, and there is a rational F -point on X (or a 0-cycle of degree 1), then $A_0(X)$ is finite.
- (b) If F is a p -adic field and X has good reduction, then $A_0(X) = 0$.
- (c) If the cohomological dimension $\text{cd } F$ of F is at most 1, then $A_0(X) = 0$.

In fact, this theorem boils down to a pure K-theoretic problem in view of the following exact sequence discovered by Bloch in paper [2]:

$$H^1(G, K_2K(X)/K_2K) \rightarrow A_0(X) \xrightarrow{\Phi} H^1(G, S(K))$$

where K is a Galois splitting field (whose definition will be given in section 1.1) for X and S is an F -torus dual to the $G := \text{Gal}(K/F)$ -module $\text{Pic } X_K$. The point is that Bloch has also shown in paper [2] that $\text{im } \Phi$ is finite in case (a) and is 0 in case (b) and (c), so theorem 0.1. will be settled as long as we know that Φ is injective, i.e., $H^1(G, K_2K(X)/K_2K) = 0$. So what remains to be shown is the following result, which holds in a more general setting.

Thm 0.2. Let E/F be a Galois extension of fields where F is a field of characteristic 0. Denote $G = \text{Gal}(E/F)$. Suppose X is a geometrically integral F -variety and there is a smooth rational F -point on X , or if $\text{cd } F \leq 1$, then

- (i) the natural map $j : K_2F(X)/K_2F \rightarrow (K_2E(X)/K_2E)^G$ is an isomorphism.
- (ii) $H^1(G, K_2E(X)/K_2E) = 0$.

In this survey, I will only focus on the result in cases (a) and (b) (corresponding to the case when there is a smooth rational F -point in theorem 0.2.) since the same proof works almost verbatim in the case (c) when $\text{cd } F \leq 1$, with only minor changes.

Rmk 0.1. Let me explain about the technical requirement that X has a smooth rational F -point. First, this is known to be true in case (b) and rational surfaces are supposed to be smooth, so indeed this theorem 0.2. implies theorem 0.1., in light of Bloch's results.

The requirement of such technical condition comes from the use of a technical tool to transfer from K_2 of a function field $K(X)$ to that of K , which is called a "specialization argument", which roughly speaking, is just to specialize an element in $K_2K(X)$ to its "value" at a smooth rational K -point, whose local ring is regular. I shall explain this in more detail in section 1.5.

Nevertheless, in the case when $\text{cd } F \leq 1$ with no smooth rational points, specialization arguments used in proof can always be substituted by an indeed easier argument to show the same intermediate

results we need, so the reason why I exclude the case when $\text{cd } F \leq 1$ is just to avoid discussions to make the proof of theorem 0.2. looks neater .

Finally, one should remark that Suslin has proved in his paper [3] that the result (i) in theorem 0.2. actually holds in general, even when there is not smooth rational F -point at hand. His proof is based on a finer result of Bloch-Kato-Gabber (see Remark 8.5.(b) of [4]) while the argument of the main part is same as that of our paper [1], which will be shown in section 2.

I shall briefly recollect some necessary background settings in section 1, including the definitions of some objects that we will use but I'm unfamiliar with personally, and the core results of Hilbert 90 for K_2 , proved in paper [5], and the so-called specialization argument. Due to the constraint of length, I will omit most of the reduction steps that are somewhat more trivial, presenting only the most important part of the proof of theorem 0.2. in section 2 and I will try to fill up some gaps in the proof (mainly because I'm not acquainted with some arguments myself).

Throughout the paper, p will always denote a prime.

§ 1. SOME BACKGROUND MATERIALS

1.1. Basic geometric settings.

In this subsection we give the basic geometric setting that we are working with.

Def 1.1. Suppose X is a scheme over a field F . X is *geometrically integral* over F if the scheme $X_{F'}$ is integral for every field extension F' of F .

Def 1.2. Suppose X is a projective surface over a field F . It is *rational* if it is smooth, geometrically integral and there exists an algebraic extension E/F such that the function field $E(X)$ of $X \times_{\text{Spec } F} \text{Spec } E$ is purely transcendental over F .

Rmk 1.1. Note that this definition differs from our usual definition of rationality. In fact, it is equivalent to saying that the F -surface X is rational in the usual sense, i.e., birationally equivalent to $\mathbb{P}_E^2$ after base change to field E .

The next definition is explained here since they appeared in the exact sequence of Bloch to finally derive theorem 0.1. about $A_0(X)$. However, since we won't present the proof of Bloch's results and what we are really concerned about is theorem 0.2., these concepts will not be actually used in later proofs.

Def 1.3. Let X and F be as above. We say F' is a *splitting field* of X if $X_{F'}$ can be obtained by blowing up a finite number of F' -points starting from $\mathbb{P}_{F'}^2$, and then blowing down a finite number of exceptional curves of the first kind defined and absolutely irreducible over F' .

Note that this is more strict than requiring $X_{F'}$ itself is rational since blowing up and down process from a rational F' -surface to $\mathbb{P}_{F'}^2$ in general case by involve blowing-up non F' -rational points. But still one can show the existence of a splitting field and taking its Galois closure gives us a Galois splitting field.

1.2. Basic definition on group cohomology.

In this subsection we will only recall the very basic definition of group cohomology (of finite groups) because the properties of group cohomology will not be used intensively, except for a restriction-inflation exact sequence and an isomorphism with Tate cohomology, which will show up and be explained during the proof process. The main reference is [6].

Suppose G is a finite group and A an abelian group equipped with a left G -action, that is to say, A is an $\mathbb{Z}[G]$ -module. We will call such A simply as a *G -module*.

Def 1.4. $A^G = \{a \in A \mid ga = a \text{ for all } g \in G\}.$

Then it is direct to check $\Gamma^G : A \mapsto A^G$ is a functor on the category of G -modules. Denotes its i -th derived functor $R^i\Gamma^G$ by $H^i(G, -)$. This is i -th *group cohomology* of G .

One (and almost the only one) nontrivial result we will need for group cohomology is the 2-periodicity for Tate cohomology groups when the group G is cyclic. In particular we would only need that $\hat{H}^{-1}(G, A) \cong H^1(G, A)$ and $\hat{H}^0(G, A) \cong H^2(G, A)$, allowing us to prove the group cohomology results via Tate cohomology involving norm maps.

The proof of these results cannot be summarized in one or two paragraphs so I shall only recall the definition of these two Tate cohomology groups during the proof. For the general theory one may refer to [10] and our results appear as definition 7.3 and proposition 14.1 therein.

1.3. Basic K-theoretic results.

In this subsection we will recall the basic definitions on K_2 groups, and then summarize the basic properties of Steinberg symbols which will be heavily used in the proof. The main reference is chapter 11-12 in [7].

Def 1.5. A *matrix unit* is a matrix $e_{ij} \in \text{GL}(R)$ over a ring R whose (i, j) -th entry is 1_R while all others being 0. An *elementary (transvection) matrix* $e_{ij}(r)$ is defined to be $I + re_{ij}$, where I is the identity matrix.

It is direct to see that the matrix units multiply as $e_{ij}e_{kl} = e_{il}\delta_{jk}$.

Then it is also easy to verify the following to relations the elementary matrices satisfy, which are also standard facts from linear algebra:

$$\begin{aligned} EL \ 1: \quad & e_{ij}(r)e_{ij}(s) = e_{ij}(r+s) \\ EL \ 2: \quad & [e_{ij}(r), e_{kl}(s)] = \begin{cases} 1 & \text{if } i \neq l, j \neq k \\ e_{il}(rs) & \text{if } i \neq l, j = k. \end{cases} \end{aligned}$$

There are also other relations, such as

$$EL \ 3: \quad (e_{12}(1)e_{21}(-1)e_{12}(1))^4 = 1.$$

In view of this, Steinberg defined the following group.

Def 1.6. Over a ring R , the *Steinberg group* $St_n(R)$ is generated by the expressions $x_{ij}(r), i \neq j, 1 \leq i \leq n, 1 \leq j \leq n$, subjecting only to relations EL1 and EL2, in which e_{ij} are replaced to be x_{ij} .

Thus if we drop the upper bound n for i, j , we get the "infinity" Steinberg group $St(R)$. And if we denote the group generated by the elementary matrices $e_{ij}(r), 1 \leq i, j \leq n, r \in R$ by $E_n(R)$ and similarly $E(R)$, we have the following surjective homomorphisms:

$$\phi_n : St_n(R) \rightarrow E_n(R), \quad \phi : St(R) \rightarrow E(R).$$

Def 1.7. $K_{2,n}(R) := \ker \phi_n, K_2(R) := \ker \phi$.

Thm 1.1. For every ring R , $K_2(R)$ is the center of $St(R)$.

In particular, this theorem implies that K_2 groups are commutative. Next we shall discuss about the elements in $K_2(R)$ more carefully. In fact, there are some special elements called the Steinberg symbols in $K_2(R)$, which turn out to generate $K_2(F)$ when $R = F$ is a field. They will be used throughout our proof.

First let us define the nearest analogue to the other two kinds of row operation matrix in linear algebra. In fact one can carry out explicit calculation to see how the following elements w_{ij} and h_{ij} behaves in the elementary matrices group.

Def 1.8. Denote the generators of $St(R)$ by $x_{ij}(r)$ as above. Then define $w_{ij}(a) = x_{ij}(a)x_{ji}(-a^{-1})x_{ij}(a)$ and $h_{ij}(a) = w_{ij}(a)w_{ij}(-1)$ for $a \in R^*$ a unit.

Prop 1.1. Suppose $a, b \in R^*$ and $ab = ba$, then for any three different positive integers i, j, k , we have

$$[h_{ik}(a), h_{ij}(b)] = h_{ij}(ab)h_{ij}(a)^{-1}h_{ij}(b)^{-1}.$$

This is a somewhat weird relation, stating that the commutator of $h_{ik}(a)$ and $h_{ij}(b)$ is independent of k . Thus we denote this commutator by $\{a, b\}_{ij}$. The next result tells more on this.

Prop 1.2. Suppose $a, b \in R^*$ and $ab = ba$, then for any two different positive integers i, j , we have $\{a, b\}_{ij} = \{a, b\}_{12}$. Also, $\{a, b\}_{ij}$ commutes with all the elements in $St(R)$, i.e., $\{a, b\}_{ij} \in Z(St(R)) = K_2(R)$.

In view of these propositions, it is well reasonable to give the following definition.

Def 1.9. Suppose R is a ring and $a, b \in R^*$, $ab = ba$. Then a *Steinberg symbol* for a, b is

$$\{a, b\} = \{a, b\}_{12} = h_{12}(ab)h_{12}(a)^{-1}h_{12}(b)^{-1} \in K_2(R).$$

Note that in the definition we require a, b to be commuting units and for our own use, it is fine to only consider R to be a field F . In this case we have the following important result, which does not hold in general by more careful analysis of the subgroups generated by w_{ij} and h_{ij} .

Thm 1.2. If F is a field, then $K_2(F)$ is generated by Steinberg symbols.

Thus it is in some sense reasonable to only study the Steinberg symbols for the study of K_2 groups of a field. Next we state the main calculational properties they satisfy that we will freely use in our proof. The proof of these properties can be simply done via linear algebraic calculations so will be omitted.

Thm 1.3. Suppose $a, b, c \in F$, where F is a field, then in $K_2(F)$, we have

- (i) $\{a, b\}^{-1} = \{b, a\}$
- (ii) $\{ab, c\} = \{a, c\}\{b, c\}$
- (iii) $\{a, bc\} = \{a, b\}\{a, c\}$
- (iv) $\{a, b\} = 1$ if $a + b = 1_F$.

Note that $\{1, b\} = 1$ for any $b \in F$ from this law by $\{a, b\} = \{a, b\}\{1, b\}$. Also note that it is this final law that $\{a, b\} = 1$ if $a + b = 1_R$ that leads to the very definition to quotient out the relations $x \otimes (1 - x)$ of Milnor K -groups.

The commutative multiplication in the K -groups will be written additively from now on. However the reader shall not forget that this really comes from multiplication in the rings. For instance, the multiplicative law for the Steinberg symbols will be written as $\{a, b\} + \{a, c\} = \{a, bc\}$ and similarly, when writing n times an element in the K -group, what we really means is taking its n -th power.

1.4. Preliminary theorems.

In this subsection we will state the key inputs to prove our theorem, including the results from Meker's and Suslin's paper [5] and a result from Bloch's paper [8].

Let's first recall the algebraic and cohomological definition about the Brauer group, both to be used later. However we shall not present the equivalence of these two definitions.

Def 1.10. A finite dimensional algebra over a field F is called a *central simple algebra (CSA)* if it is *central*, i.e., has the center exactly F , and *simple*, i.e., has no non-trivial two-sided ideals. Two CSAs A, B over F are *Brauer equivalent* if there exist m, n such that the matrix algebras $M_n(A) \cong M_n(B)$.

One can check that the set of such Brauer equivalence classes form a group, called the *Brauer group*, denoted by $\text{Br}(F)$, whose multiplication is given by tensor product over F , and $M_n(F)$ being identity, $[M_n(A^{\text{op}})]$ being inverse of $[M_n(A)]$.

For field extensions E/F , extension by scalar $A \mapsto A \otimes_F E$ induces a homomorphism $\phi : \text{Br}(E) \rightarrow \text{Br}(F)$, and $\ker \phi$ is the *relative Brauer group*, denoted by $\text{Br}(E/F)$.

Equivalently, we have the definition for Brauer groups and relative groups by group cohomology where F_{sep} is the separable closure in a chosen algebraic closure, which means to take limit of the relative version:

$$\begin{aligned}\text{Br}(E/F) &= H^2(\text{Gal}(E/F), E^*) \\ \text{Br}(F) &= H^2(\text{Gal}(F_{\text{sep}}/F), F_{\text{sep}}^*).\end{aligned}$$

Finally, there is a special kind of central simple algebra that we shall explain here for the statement of the theorem below.

Def 1.11. Suppose F is a field containing a n -th root of unity ζ , then the cyclic algebra $(a, b)_{\zeta}$ is the associative algebra over F subjecting to the relations

$$x^n = a, \quad y^n = b, \quad yx = \zeta xy.$$

There is a well-defined map (though we won't check this) $K_2F/nK_2F \rightarrow \text{Br } F : \{a, b\} \mapsto [(a, b)_{\zeta}]$, called the *norm residue map*.

Suppose F is a field containing a p -th root of unity ζ and $p \neq \text{char } F$. The first result gives a characterization of K_2F/pK_2F while the second result, on the contrary, characterizes the p -torsion part pK_2F . (For reference, see [5], 9.4, 11.4, 11.5 for the first and 10.4 for the second.)

Thm 1.4 (MS a). The norm residue symbol $K_2F/pK_2F \rightarrow \text{Br } F : \{a, b\} \mapsto [(a, b)_{\zeta}]$, is an isomorphism onto the p -torsion subgroup $\text{Br}_p(F)$ of the Brauer group $\text{Br}(F)$.

Thm 1.5 (MS c). Any p -torsion element of K_2F may be written as a Steinberg symbol $\{a, \zeta\}$ for some $a \in F^*$.

In this survey, the way we really use *MS a* is via the following. Suppose X is a "good enough" variety (for the actual condition we require on X , see section 2) and $E = F(b)$ is a cyclic Galois extension with Galois group G , and $b^p = a \in k^*$. Then we have the following:

Prop 1.3. Any element in the kernel of the natural map $K_2F(X)/pK_2F(X) \rightarrow K_2E(X)/pK_2E(X)$ may be represented by a symbol $\{a, f\}$ for some $f \in F(X)^*$.

Proof. It is easy to see this from the cohomological definition of the Brauer group. For the definition of the relative Brauer group, we find the following short exact sequence:

$$0 \rightarrow \text{Br}(E(X)/F(X)) = H^2(G, E(X)^*) \xrightarrow{\psi} \text{Br}(F(X)) \rightarrow \text{Br}(E(X)).$$

Here ψ associates $f \in F(X)^*$ to the class of the class of cyclic algebra $(a, f)_{\zeta}$. Since G is cyclic, we have the 2-periodicity for the Tate cohomology groups, in particular implying that $H^2(G, E(X)^*) \cong \hat{H}^0(G, E(X)^*) = F(X)^*/N_{E(X)/F(X)}(E(X)^*)$ by definition.

In fact, since the extension is cyclic of order p , the relative Brauer group is p -torsion since we can see that every element is annihilated by p via taking norm map. (For a similar argument, one may check the proof for step 2.1 below.)

Thus the Brauer groups in the exact sequence above can be replaced by their p -torsion parts, which can be further identified by K_2/p , in light of theorem *MS a*, i.e., we have

$$0 \rightarrow F(X)^*/N_{E/F}(E(X)^*) \xrightarrow{\psi} K_2F(X)/pK_2F(X) \rightarrow K_2E(X)/pK_2E(X).$$

Hence the kernel that we are looking for is just $\psi(F(X)^*/N_{E/F}(E(X)^*))$, which consists of the class of cyclic algebras $(a, f)_\zeta$, which is in turn identified with $\{a, f\}$ via theorem *MS a*, and we are done. $\square$

Finally, we have the "Hilbert 90" for K_2 , which appears in [5] as theorem 14.1. (In general, we have a norm map $N_{E/F}$ acting on higher (Milnor) K -groups, however the construction of it using Bass-Tate construction is too complicated to write down so I will omit here.)

Thm 1.6 (MS b). Let E/F be a cyclic extension of degree n , where $(n, \text{char } F) = 1$. Suppose σ is a generator of $\text{Gal}(E/F)$. Then $\ker(N_{E/F} : K_2E \rightarrow K_2F) = (1 - \sigma)K_2F$.

(The naming of the theorems *MS b* and *MS c* are kept the same as in paper [1] but I reverse their order in this survey for coherence since I think *MS a* and *MS c* both discuss about the p -torsion part of K_2F .)

Finally, we would need the following result from Bloch's paper [8]. We state it in the general form that can be derived from the case where X is a curve (and it is this case that Bloch has proven in his paper) as in paper [1] for later use. It shall need some work to deduce this general case from the curve case by a nontrivial induction, nevertheless, we will omit the proof here.

Thm 1.7 (B). Let F be a field with $\text{char } k = 0$. Suppose $\bar{F}$ is the algebraic closure of F and X is a geometrically integral F -variety. Let ζ be a primitive n -th root of unity in $\bar{F}$ and $f \in F^*(X)^*$ such that $\{f, \zeta\} = 0 \in K_2\bar{F}(X)$, then there exists $\alpha \in F^*, g \in F(X)^*$ such that $f = \alpha \cdot g^n$.

1.5. Specialization arguments.

Finally, in this subsection, we shall introduce an important but technical process that will be used throughout the proof. Roughly speaking, this allows us to transfer from $K_2F(X)$ to K_2F when F is a field and X has a rational F -point P , by specializing a function on $F(X)$ to its value at the point P . We will start from its rigorous definition as in [5].

If O is a discrete valuation ring with field of fractions Q and residue field k , and local parameter denoted by π . Then there is a canonical homomorphism $s_\pi : K_2(Q) \rightarrow K_2(k)$ defined by

$$s_\pi(\{x, y\}) = \left\{ \left(\frac{x}{\pi^{\text{ord}(x)}} \right), \left(\frac{y}{\pi^{\text{ord}(y)}} \right) \right\}.$$

Here the overline is the rigorous notation for the corresponding class in the residue field $k \cong O/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal. Practically, one may think of O as the function field and k as the base field of a k -variety (which will be exactly our case when we come to the discussion of the regular local ring below), and then x here is a rational function f . Suppose $f = x^n \tilde{f}$ where $\tilde{f}$ is invertible, then $\left(\frac{x}{\pi^{\text{ord}(x)}} \right)$ is just the constant term of $\tilde{f}$, i.e., value of $\tilde{f}$ at $\mathfrak{o}$. For instance, for $f = \frac{5t^2+9t}{3t^3+6t+1}$, this value is 9 while for $f = \frac{1}{t(1-t)}$ this value is 1.

Now we come to the general case for regular local ring O with field of fractions Q and residue field k . Suppose $\pi = (\pi_1, \dots, \pi_r)$ is a regular system of parameters and we denote by Q_i the field of fractions of the ring $O/(\pi_1, \dots, \pi_i)$, then iterating the process as above, we get the homomorphism:

$$s_\pi : K_2(Q) \xrightarrow{s_{\pi_1}} K_2(Q_1) \xrightarrow{s_{\pi_2}} K_2(Q_2) \xrightarrow{s_{\pi_3}} \dots \xrightarrow{s_{\pi_r}} K_2(Q_r) = K_2(k).$$

This map is certainly non-canonical and depends on the choice of the system π . Thinking practically as above in multivariable formal power series ring $R = k[[t_1, \dots, t_d]]$, this boils down to the fact that

this ring is not a valuation ring in general since the natural valuation on the multivariable Laurent series ring $Q = k((t_1, \dots, t_n))$ does not fit for R . (For instance, $\frac{t_1}{t_2}$ has valuation 0 in Q but is not an element in R .)

Now let us come back to the geometric setting. Suppose X is a K -variety. Then a rational K -point P has local ring O whose residue field is K and X being smooth at P implies that O is regular, and the quotient field of the local ring O is exactly the global rational function field $K(X)$ (on an integral scheme in general). These ensures us to apply the previous setting. Hence a specialization argument is just to give a transfer map from $K_2K(X)$ to K_2K . Also note that different chosen π may give different specialization maps as explained above, but we don't need it to be unique! We only need the existence of such a transfer map.

However, as in the practical example above, for an element which belongs to the local ring O (thought as the multivariable formal power series ring R) itself, the valuation is reasonable anyway. For a general specialization one may easily see similarly that the restriction of a specialization map s_π to $K_2(O)$ is the same as the natural homomorphism $K_2(O) \rightarrow K_2(k)$ induced by the map $O \rightarrow k$. It is in this sense that we roughly think specialization of $f \in K(X)$ as a process to "evaluate" at a certain point P and denote it (with abuse of notation) by $f(P)$ in the proof.

§ 2. PROOF (SKETCH) OF THE MAIN THEOREM

The statement of the proof is in section 0 and will not be repeated here.

The whole proof of this theorem is divided into several steps. I shall not present the full proof. In fact, most part of the proof is almost routine reduction. For instance, it is direct to reduce the general case to the case of finite Galois extension and the latter to the case where the Galois group $\text{Gal}(E/F)$ is a finite p -group. Of, course, it is not hard to imagine if we have proved the theorem when $\text{Gal}(E/F)$ is a cyclic group of order p , we can carry out an induction to further prove the previous case, which we shall show in detail later to exhibit certain necessity to prove statement (i) first, though our original point of interest lies in statement (ii).

On the other hand, in the final case which we have reduced to above, by adding a primitive p -th root of unity and some simple arguments, we may assume F has a primitive p -th root of unity ζ and E/F is cyclic of prime degree p . To prove the theorem in this case is the most nontrivial part of the whole proof that we shall demonstrate below.

We shall divide the proof in this case further into three steps. From now on, suppose X is a geometrically integral F -variety, where F is a field of characteristic 0 and X has a smooth rational F -point and suppose further that F contains a primitive p -th root of unity ζ and E/F is a cyclic extension of degree p . This will be referred to "our case" below.

Step 2.1. In our case, for any algebraic extension E/F , the natural map $j : K_2E(X)/K_2E \rightarrow K_2F(X)/K_2F$ is injective.

Rmk 2.1. This step actually holds without extra assumption on F and the extension E/F is only required to be algebraic. However, it is also routine to reduce the general setting to our setting, using the same set of tools as in the reduction of the main theorem, so this lemma will also be proved only in this special situation.

Proof. Suppose $z \in K_2F(X)$ such that $z|_{K_2E(X)} = \beta|_{K_2E(X)}$ where $\beta \in K_2E$. Taking norm map $N_{E(X)/F(X)}$ on both sides, viewing E as constant function in $E(X)$ and we get that $pz = \gamma$, where $\gamma \in K_2F$. (Also note that for a geometrically integral variety X such that $K(X) \cong k(X) \otimes_k K$ and the extension degree of $E(X)/F(X)$ is same to that of E/F , which is p here.)

If X has a smooth F -point, specialize this formula to this point and we may assume $\gamma = p\delta$, where $\delta \in K_2F$ as well. Thus $z - \delta$ is in the p -torsion part of $K_2F(X)$. By *MS* c, we may find $f \in F(X)^*$ such that

$$z - \delta = \{f, \zeta\} \in K_2F(X).$$

Since $z|_{K_2E(X)} = \beta|_{K_2E(X)}$ and $\beta \in K_2E$, also $\delta \in K_2F \subseteq K_2E$, so does $\{f, \zeta\}$. Passing over to $\overline{F} = \overline{E}$ (and the corresponding function field), we find that $\{f, \zeta\} \in K_2\overline{F}(X)$ comes from $K_2\overline{E}$. Indeed this implies that $\{f, \zeta\} = 0$ by specialization to a smooth $\overline{F}$ -point P , since $\zeta^p = 1$ and $\{f(P), \zeta\} = \{f(P)^{\frac{1}{p}}, \zeta\}^p = f(P)^{\frac{1}{p}}, \zeta^p = 0$. Hence by theorem B we find that there exist $\alpha \in F^*, g \in F(X)^*$ such that $f = \alpha \cdot g^p$. Thus $\{f, \zeta\} = \{\alpha, \zeta\}\{g^p, \zeta\} = \{\alpha, \zeta\}\{g, \zeta^p\} = \{\alpha, \zeta\}$, coming from $K_2F \subseteq K_2F(X)$, so does z since δ itself belongs to K_2F . $\square$

Step 2.2. In our case, (ii) of the theorem 0.2. holds.

Rmk 2.2. In fact this result holds without extra assumption on F and the extension E/F is only required to be cyclic, we shall give the proof in this more general case.

Proof. For the proof of this step and later, we need the following commutative triangle as an analogue of the usual push-pull formula, proved by Bloch in [9]:

$$\begin{array}{ccc} K_2E(X) & \xrightarrow{N} & K_2E(X) \\ & \searrow N_2 & \nearrow i \\ & K_2F(X) & \end{array}$$

Here i is the inclusion induced by $F(X) \subseteq E(X)$, N_2 is the norm map, and $N = \sum_{g \in G} g$. Since this same N restricts to a map $K_2E \rightarrow K_2F$, we may write a similar diagram for field extension E/F . Then we get the following induced commutative triangle:

$$\begin{array}{ccc} K_2E(X)/K_2E & \xrightarrow{\tilde{N}} & K_2E(X)/K_2E \\ & \searrow N_3 & \nearrow j \\ & K_2F(X)/K_2F & \end{array}$$

Here $j, N_3, \tilde{N}$ are the induced version of the inclusion i , the norm N_2 and the map N above, respectively. By Step 2.1., the map j here is injective. So we have $\ker N_3 = \ker \tilde{N}$.

Now consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_2E & \longrightarrow & K_2E(X) & \longrightarrow & K_2E(X)/K_2E \longrightarrow 0 \\ & & \downarrow N_1 & & \downarrow N_2 & & \downarrow N_3 \\ 0 & \longrightarrow & K_2F & \longrightarrow & K_2F(X) & \longrightarrow & K_2F(X)/K_2F \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & K_2F/N_1(K_2E) & \xrightarrow{\theta} & K_2F(X)/N_2(K_2E(X)) & & \end{array}$$

Here the inclusion of $K_2E \hookrightarrow K_2E(X)$ comes from viewing an element in E as a constant function in $E(X)$, and similar for $K_2F \hookrightarrow K_2F(X)$. Hence the norm N_2 on $K_2E(X)$ naturally restricts to N_1 on $K_2E \subseteq K_2E(X)$, obtaining the natural map θ . When there is smooth rational F -point P on X , the specialization map s_P gives an left inverse to θ , hence in this case θ must be injective.

Thus by snake lemma we get a surjection $\ker N_2 \twoheadrightarrow \ker N_3 = \ker \tilde{N}$, as argued above. By Theorem MS b, we know that the elements in $\ker N_2$ are of the form $\sigma(y) - y, y \in K_2E(X)$, and σ is a generator of G . This shows exactly that if $\tilde{N}(x) = 0$, then $x = \sigma(y) - y$ for some $y \in K_2E(X)$. By the definition of Tate cohomology, $\hat{H}^{-1}(G, K_2E(X)/K_2E) = 0$ and by the periodic isomorphism $\hat{H}^n(G, M) \cong \hat{H}^{n+2}(G, M)$ for cyclic group G , we see that $H^1(G, K_2E(X)/K_2E) = \hat{H}^1(G, K_2E(X)/K_2E) \cong \hat{H}^{-1}(G, K_2E(X)/K_2E) = 0$ and we are done. $\square$

Step 2.3. In our case, (i) of the theorem 0.2. holds.

Proof. By step 2.1., the map j is injective. And suppose now $\alpha \in K_2E(X)$ whose class $\bar{\alpha} \in K_2E(X)/K_2E$ is invariant under action of G . Throughout the proof, since we want to prove that $\bar{\alpha} = j(x)$ for certain $x \in K_2F(X)/K_2F$, we might freely change α up to an element in $j(K_2F(X))$ and up to an element in K_2E .

The first step in the proof is to represent α by certain Steinberg symbol modulo K_2E and $K_2F(X)$ as explained above, to further carry out the calculation. From the second commutative triangle as shown in step 2.1., since α is G -invariant, we have $p\alpha = \beta + \delta$, where $\beta \in K_2F(X), \delta \in K_2E$. By specialization (and base change to E) to a smooth rational F -point P , we get $p\alpha(P) = \beta(P) + \delta$, where $\alpha(P), \delta \in K_2E$ and $\beta(P) \in K_2F$. Thus $p\alpha = \beta$ by change α to $\alpha - \alpha(P)$ and β to $\beta - \beta(P)$.

Now $\beta \in K_2F(X)$ which is a p -th power of an element in $K_2E(X)$. Suppose $\beta = p\gamma + \epsilon$ where $\gamma \in K_2F(X)$ and $\epsilon = \beta - p\gamma$ is the error term. Then indeed ϵ lies in the kernel of the map $K_2F(X)/pK_2F(X) \rightarrow K_2E(X)/pK_2E(X)$. Hence by the reformulation of theorem *MS a* as explained in section 1.4., ϵ might be represented a Steinberg symbol $\{a, f\}$ for some $f \in F(X)^*$ and $a \in F^*$, satisfying $E = F(b), b^p = a$. Summarizing, we have that

$$p(\alpha - \gamma) = \{a, f\} = \{b^p, f\} = p\{b, f\}.$$

Since $\gamma \in K_2F(X)$, we might change α to $\alpha - \gamma$ and we get that $\alpha - \{b, f\}$ is a p -torsion in $K_2F(X)$. Hence by theorem *MS c*, we obtain certain $g \in F(X)^*$ (also noticing that $\{g^{-1}, \zeta\} = -\{\zeta, g^{-1}\} = \{\zeta, g\}$) such that

$$\alpha = \{b, f\} + \{\zeta, g\}.$$

This is the starting point to analyze by calculating with these Steinberg symbols. Now suppose σ is a generator of G such that $\sigma(b) = \zeta b$. Since $\zeta \in F$ and $f \in F(X)^*$, they are invariant under G . Hence we get that

$$\sigma(\alpha) = \sigma(\{b, f\} + \{\zeta, g\}) = \{\sigma(b), \sigma(f)\} + \{\sigma(\zeta), \sigma(g)\} = \{\sigma(b), f\} + \{\zeta, \sigma(g)\}.$$

Then since the class of α is G -invariant, we get

$$\sigma(\alpha) - \alpha = \left\{ \frac{\sigma(b)}{b}, f \right\} + \left\{ \zeta, \frac{\sigma(g)}{g} \right\} = \left\{ \zeta, f \cdot \frac{\sigma(g)}{g} \right\} = 0 \in K_2E(X)/K_2E.$$

By base change to the algebraic closure $\bar{E} = \bar{F}$, we can again argue by specialization argument as in the end of the proof of step 2.1. that $\{\zeta, f \cdot \frac{\sigma(g)}{g}\}$ itself is equal to $0 \in K_2\bar{F}(X)$ (because it comes from K_2F and by specialization to a smooth rational $\bar{F}$ -point P , we have that $\{\zeta, (f \cdot \frac{\sigma(g)}{g}(P))^{\frac{1}{p}}\}^p = \{\zeta, f \cdot \frac{\sigma(g)}{g}(P)\} = \{\zeta^p, (f \cdot \frac{\sigma(g)}{g}(P))^{\frac{1}{p}}\} = 0$). Thus by the theorem *B*, we know that there exists $\lambda \in E^*$ and $h \in E(X)^*$ such that

$$f \cdot \frac{\sigma(g)}{g} = \lambda \cdot h^p.$$

By taking norms $N_{E/F}$ we find that $f^p = N(\lambda)N(h)^p$ since $N(\frac{\sigma(g)}{g}) = 1$ and $f \in F(X)^*$, i.e., $(\frac{f}{N(h)})^p = N(\lambda) \in F^*$.

There are two ways to settle the proof from here, the one using heavier specialization arguments while the other using a more ingenious observation. We shall only introduce the latter since it is this proof that Suslin used in his paper [3] to generalize the result to the general case without a smooth rational F -point.

Now suppose $\frac{f}{N(h)} = \mu \in E^*$, then $\mu^p = N(\lambda) \in F^*$ and $N(\frac{\mu}{\lambda}) = 1$, hence by the usual Hilbert 90, there exists $\rho \in E^*$ such that $\frac{\mu}{\lambda} = \frac{\sigma(\rho)}{\rho}$. Then we have

$$\mu^{-1} f \cdot \frac{\sigma(\rho g)}{\rho g} = \mu^{-1} \cdot \frac{\mu}{\lambda} \cdot \lambda \cdot h^p = h^p.$$

Since $\mu, \rho \in E^*$, changing f to $\mu^{-1}f$ adds $\{b, \mu^{-1}\} \in K_2E^*$ to α and doesn't change the class $\bar{\alpha} \in K_2E(X)/K_2E$, and similar for changing g to $\rho(g)$. Thus we might suppose that $f = N(h)$ and $N(h) \frac{\sigma(g)}{g} = h^p$. The key observation here is that the latter formula is equivalent to that the function

$$t = \frac{g}{\prod_{i=1}^{p-1} (\sigma^i(h))^i}$$

being invariant under σ , i.e., $t \in F(X)^*$. This comes from direct calculation since

$$\frac{\sigma(t)}{t} = \frac{\sigma(g)}{g} \frac{\prod_{i=1}^{p-1} (\sigma^i(h))^i}{\prod_{i=2}^p (\sigma^i(h))^{i-1}} = \frac{\sigma(g)}{g} \frac{(\sigma^p(h))^{p-1}}{\prod_{i=1}^{p-1} \sigma^i(h)} = \frac{\sigma(g)}{g} \frac{h^p}{N(h)}.$$

Thus we have

$$\{\zeta, g\} = \{\zeta, t\} + \sum_{i=1}^{p-1} \{\zeta, (\sigma^i(h))^i\} = \{\zeta, t\} + \sum_{i=1}^{p-1} \{\zeta^i, \sigma^i(h)\}.$$

Since $\{\zeta, t\} \in K_2F(X)^*$, we can change α to $\alpha - \{\zeta, t\}$ and now we have

$$\begin{aligned} \alpha &= \{b, f\} + \{\zeta, g\} - \{\zeta, t\} = \{b, N(h)\} + \{\zeta, g\} - \{\zeta, t\} \\ &= \sum_{i=0}^{p-1} \{b, \sigma^i(h)\} + \sum_{i=1}^{p-1} \{\zeta^i, \sigma^i(h)\} \\ &= \sum_{i=0}^{p-1} \{b\zeta^i, \sigma^i(h)\} = \sum_{i=0}^{p-1} \sigma^i(\{b, h\}) \\ &= \sum_{g \in G} g(\{b, h\}) = j \circ N_3(\{b, h\}). \end{aligned}$$

We use in the last two lines that $\sigma(b) = \zeta b$ and the second commutative triangle. Hence we are done. $\square$

Finally, we shall also exhibit the induction step to indicate how we use (ii) in the reduction process to show (i).

Step 2.4. (i) and (ii) hold when $G = \text{Gal}(E/F)$ is a finite p -group.

Proof. We have said at the beginning of this section that the case when F may not have a primitive p -th root of unity can be easily reduced to the case we have already dealt with. So suppose we know this result holds when $G = \mathbb{Z}_p$ and we will prove the general case by induction on n , where $\#G = p^n$. Suppose the result holds for p -groups of order $\leq p^{n-1}$ and $\#G = p^n$.

By the general result on p -groups in group theory we know the existence of a normal subgroup H with order p of G and set $T = G/H$ and $L = E^H$. Then by Galois theory, $L = \text{Gal}(L/F)$. We need the following restriction-inflation sequence from general group cohomology theory for induction:

$$0 \rightarrow H^1(T, (K_2E(X)/K_2E)^H) \rightarrow H^1(G, K_2E(X)/K_2E) \rightarrow H^1(H, K_2E(X)/K_2E),$$

where we know the last term vanishes by step 2.4., and we need step 2.3. to identify $(K_2E(X)/K_2E)^H$ with $K_2L(X)/K_2L$. (This is the reason that we need (i) in this proof to finish.) Then by induction

hypothesis $H^1(T, (K_2E(X)/K_2E)^H) = H^1(T, K_2L(X)/K_2L)$ vanishes, hence $H^1(G, K_2E(X)/K_2E)$ so (ii) is proved. But (i) is also direct to see via the inclusions

$$K_2F(X)/K_2F \hookrightarrow K_2L(X)/K_2L \hookrightarrow K_2E(X)/K_2(E),$$

and by induction hypothesis we know

$$K_2F(X)/K_2F \cong (K_2L(X)/K_2L)^T \cong ((K_2E(X)/K_2E)^H)^T \cong (K_2E(X)/K_2E)^G.$$

□

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