

# POISSON DEFORMATION OF DEL PEZZO SURFACES

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ABSTRACT. We study Poisson deformations of the simple elliptic surface singularities of types  $\tilde{E}_6$ ,  $\tilde{E}_7$ , and  $\tilde{E}_8$ . Their classical semiuniversal deformations are realized by families of affine del Pezzo surfaces. We explain how these families admit a natural Poisson enhancement: adjoining one parameter that rescales the Jacobian Poisson bracket gives a semiuniversal Poisson deformation. The proof is based on a computation of the Poisson cohomology of the corresponding singular hypersurface algebra, using a differential graded resolution and an associated spectral sequence.

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## 1. INTRODUCTION

Del Pezzo surfaces form one of the basic classes of algebraic surfaces. Let  $S$  be a projective del Pezzo surface and let  $E \in |-K_S|$  be a smooth anticanonical divisor. By the adjunction formula,  $E$  is an elliptic curve, while the complement

$$X := S \setminus E$$

is an affine surface. Moreover, a nonzero section of the anticanonical bundle determines a Poisson structure on  $S$ , whose restriction to  $X$  is symplectic.

For del Pezzo surfaces of type  $E_\ell$ , where  $\ell = 6, 7, 8$ , the affine surface  $X$  admits an embedding into  $\mathbb{C}^3$ . Its defining equation can be written in the form

$$\phi_{P,Q,R}^{\tau,\nu}(x, y, z) = \tau xyz + P(x) + Q(y) + R(z) + \nu,$$

where

$$P(x) = \frac{x^p}{p} + \alpha_1 x^{p-1} + \cdots + \alpha_{p-1} x,$$

and similarly for  $Q(y)$  and  $R(z)$ . The triples  $(p, q, r)$  are

$$(3, 3, 3), \quad (4, 4, 2), \quad (6, 3, 2)$$

in types  $E_6, E_7, E_8$ , respectively.

When the lower-order parameters vanish, the equation

$$\phi^\tau(x, y, z) = \tau xyz + \frac{x^p}{p} + \frac{y^q}{q} + \frac{z^r}{r} = 0$$

defines an isolated simple elliptic singularity of type  $\tilde{E}_\ell$ . A classical result states that the family obtained by varying the coefficients of  $P, Q, R$ , together with  $\tau$  and  $\nu$ , is a flat  $\mu$ -parameter family of affine del Pezzo surfaces and gives a semiuniversal deformation of the corresponding simple elliptic singularity. Here

$$\mu = \dim_{\mathbb{C}} \mathbb{J}(\phi^\tau) = p + q + r - 1$$

is the Milnor number. We recall this result in [theorem 3.8](#); see [\[EG10, §§2.4–2.5, especially Theorem 2.5.3\(ii\)\]](#), whose proof goes back to a more general setup, see e.g. [\[Loo84, Chapter 6\]](#).

The same equations carry additional Poisson-geometric information. Indeed, every polynomial  $\phi \in \mathbb{C}[x, y, z]$  determines the Jacobian Poisson bracket

$$\{x, y\}_\phi = \frac{\partial \phi}{\partial z}, \quad \{y, z\}_\phi = \frac{\partial \phi}{\partial x}, \quad \{z, x\}_\phi = \frac{\partial \phi}{\partial y}.$$

The polynomial  $\phi$  is a Casimir for this bracket, so the quotient

$$\mathcal{B}_\phi := \mathbb{C}[x, y, z]/(\phi)$$

inherits a Poisson algebra structure. Thus, the classical deformation of the hypersurface  $\phi^\tau = 0$  naturally gives a family of Poisson algebras.

The main result discussed in this paper is [theorem 4.1](#). It states that the family

$$\left\{ \mathcal{B}_\phi \mid \phi = c \phi_{P,Q,R}^{\tau,\nu}, \quad c \in \mathbb{C}^\times \right\}$$

is a semiuniversal Poisson deformation of  $\mathcal{B}_{\phi^\tau}$ . This is the Poisson enhancement of the classical result in [theorem 3.8](#). After forgetting the Poisson bracket, one recovers the classical  $\mu$ -parameter family of affine del Pezzo surfaces. The additional parameter  $c$  does not change the hypersurface  $\{\phi = 0\}$ , but it rescales the induced Poisson bracket. Consequently, the Poisson deformation space has  $\mu + 1$  independent infinitesimal directions. This result is due to Etingof and Ginzburg; see [\[EG10, Theorem 2.5.3\(iii\) and Remark 2.5.5\]](#).

**1.1. Deformations and cohomology.** We briefly recall the deformation-theoretic ideas used in this paper; see also [\[EG10, §2.1\]](#). The results will not be summarized in their precise forms since they are too deep to prove and even to state. We merely want to emphasize some philosophy in this subsection and introduce the definition of a semiuniversal deformation.

Let  $A$  be an algebraic object, such as a commutative algebra or a Poisson algebra. Informally, a deformation of  $A$  is a flat family  $\mathcal{A}$  over a pointed parameter space  $(S, 0)$  whose fiber over 0 is  $A$ .

Flatness expresses the condition that the members of the family vary continuously in an algebraic sense and that no unexpected relations appear or disappear under specialization.

The simplest deformations are first-order deformations, parametrized by the algebra of dual numbers

$$\mathbb{C}[\varepsilon]/(\varepsilon^2).$$

For example, a first-order deformation of a Poisson bracket has the form

$$\{f, g\}_\varepsilon = \{f, g\} + \varepsilon \beta(f, g), \quad \varepsilon^2 = 0.$$

The Leibniz and Jacobi identities, considered modulo  $\varepsilon^2$ , impose linear conditions on  $\beta$ . These conditions say precisely that  $\beta$  is a cocycle in the appropriate Poisson deformation complex. Two infinitesimal deformations that differ by an infinitesimal change of coordinates determine cohomologous cocycles. Thus, equivalence classes of first-order Poisson deformations are described by the second Poisson cohomology group

$$PH^2(A).$$

More generally, the usual principle of deformation theory is that second cohomology parametrizes infinitesimal deformation directions, whereas third cohomology contains the obstructions to extending a first-order deformation to higher orders. In the Poisson setting, this role is played by  $PH^2(A)$  and  $PH^3(A)$ , respectively.

Let  $\mathcal{A} \rightarrow S$  be a deformation of an object  $A = A_0$  over a pointed base  $(S, 0)$ . A tangent vector  $v \in T_0S$  may be represented by a morphism

$$v: \operatorname{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2)) \rightarrow S$$

which maps the closed point to 0. Pulling back  $\mathcal{A}$  along  $v$  gives a first-order deformation of  $A$ . Taking its deformation class defines the *Kodaira–Spencer map*

$$\operatorname{KS}_0: T_0S \rightarrow H^2(A), \quad v \mapsto [v^*\mathcal{A}].$$

Here  $H^2(A)$  denotes the second cohomology group governing the deformation problem under consideration. In particular, for deformations of a Poisson algebra  $A$ , the relevant group is its second Poisson cohomology group, so that we get the corresponding Kodaira–Spencer map

$$\operatorname{KS}_0: T_0S \rightarrow PH^2(A).$$

Thus, the Kodaira–Spencer map measures the infinitesimal variation of the family. Its kernel consists of tangent directions along which the deformation is infinitesimally trivial, while its image consists of the first-order deformation classes realized by the given family.

In the smooth-base setting considered here, the deformation  $\mathcal{A} \rightarrow S$  is called *semiuniversal* if its Kodaira–Spencer map is an isomorphism:

$$\operatorname{KS}_0: T_0S \xrightarrow{\sim} H^2(A),$$

or, in the Poisson case,

$$\operatorname{KS}_0: T_0S \xrightarrow{\sim} PH^2(A).$$

Surjectivity means that every first-order deformation of  $A$  occurs in the family, whereas injectivity means that the base has no redundant first-order parameters. The prefix “semi” reflects the fact that a morphism from another deformation to the semiuniversal family need not be unique. Therefore, a semiuniversal deformation may be viewed intuitively as a smallest family containing all infinitesimal deformation directions of  $A$ .

**1.2. Organization of the paper.** In [section 2](#), we review polyvector fields, contraction operators, generalized Lie derivatives, the Schouten bracket, and Poisson bivectors. These identities provide the algebraic framework for the Poisson cohomology computations used later.

In [section 3](#), we recall the geometry of del Pezzo surfaces, quasi-homogeneous surface singularities, and simple elliptic singularities. We also explain how affine del Pezzo surfaces arise as deformations of the corresponding elliptic cones and state the classical semiuniversal deformation theorem.

Finally, in [section 4](#), we study the Poisson cohomology of the singular hypersurface algebra  $\mathcal{B}_\phi$ . Using a differential graded resolution, we construct a spectral sequence computing this cohomology. Together with Pichereau's calculation of the relevant Lichnerowicz–Koszul–Brylinski cohomology, this yields the description of  $PH^2(\mathcal{B}_{\phi^\tau})$  and proves the main theorem.

## 2. POLYVECTOR FIELDS, SCHOUTEN BRACKETS AND POISSON STRUCTURES

We will introduce in this section polyvector fields over an algebra, equipped with a bracket called Schouten bracket. One important result that will be proven in this section is that vanishing Schouten bracket induces Poisson structure, namely [proposition 2.8](#), which will in particular explain the Poisson structure on del Pezzo surfaces, to be introduced in next section. Besides, polyvector fields will be used in [section 4](#) to calculate the Poisson cohomology.

Throughout this section,  $A$  is a commutative unital  $\mathbb{C}$ -algebra, which we do not assume to be smooth in general. Let  $\Omega_A^1$  be the module of Kähler differentials and put

$$\Omega_A^q := \bigwedge_A^q \Omega_A^1, \quad \Omega_A^\bullet := \bigoplus_{q \geq 0} \Omega_A^q.$$

The universal derivation  $d: A \rightarrow \Omega_A^1$  extends uniquely to the de Rham differential  $d: \Omega_A^q \rightarrow \Omega_A^{q+1}$ .

### 2.1. Polyvector fields and contraction.

**Definition 2.1** (Polyvector fields). For  $p \geq 1$ , let

$$\text{AltDer}_{\mathbb{C}}^p(A, A)$$

denote the space of alternating  $\mathbb{C}$ -multilinear maps  $A^p \rightarrow A$  that are derivations in each argument. We define

$$\mathfrak{X}^p(A) := \text{AltDer}_{\mathbb{C}}^p(A, A) \cong \text{Hom}_A(\Omega_A^p, A), \quad \mathfrak{X}^0(A) := A, \quad \mathfrak{X}(A)^\bullet := \bigoplus_{q \geq 0} \mathfrak{X}^q(A).$$

The displayed isomorphism sends  $X \in \text{Hom}_A(\Omega_A^p, A)$  to the multiderivation

$$(a_1, \dots, a_p) \mapsto X(da_1 \wedge \dots \wedge da_p).$$

Its inverse follows from the universal property of Kähler differentials and of the exterior power. An element of  $\mathfrak{X}^p(A)$  is called a  $p$ -polyvector field.

**Remark 2.2.** This definition is valid even when  $A$  is singular. In particular, we do not identify  $\mathfrak{X}^p(A)$  with  $\bigwedge_A^p \text{Der}_{\mathbb{C}}(A) \cong \bigwedge_A^p \text{Hom}_A(\Omega_A^1, A)$ . In smooth case these two algebras are indeed identified, but we insist on our [definition 2.1](#) in this paper.

One of the problems one should pay attention to is that under [definition 2.1](#), there does not *a priori* exist a wedge product  $\wedge: \mathfrak{X}^p A \times \mathfrak{X}^q A \rightarrow \mathfrak{X}^{p+q} A$ , unlike the case where we have a natural wedge product  $\bigwedge_A^p \text{Hom}_A(\Omega_A^1, A) \times \bigwedge_A^q \text{Hom}_A(\Omega_A^1, A) \rightarrow \bigwedge_A^{p+q} \text{Hom}_A(\Omega_A^1, A)$ .

Indeed, this is one of the main reasons that we introduce the contraction operators below. Roughly speaking, one can define wedge product in our case as the adjoint of contraction operator. But we do not really use this in later sections so this will not be discussed in details.

We now define contraction without choosing a basis and without writing a polyvector as a sum of decomposable polyvectors. For  $q \geq p$ , let

$$\Delta_{p,q-p}: \Omega_A^q \longrightarrow \Omega_A^p \otimes_A \Omega_A^{q-p}$$

be the exterior coalgebra map given by

$$\begin{aligned} \Delta_{p,q-p}(\alpha_1 \wedge \cdots \wedge \alpha_q) &= \sum_{\sigma \in \text{Sh}(p,q-p)} \text{sgn}(\sigma) (\alpha_{\sigma(1)} \wedge \cdots \wedge \alpha_{\sigma(p)}) \\ &\quad \otimes (\alpha_{\sigma(p+1)} \wedge \cdots \wedge \alpha_{\sigma(q)}), \end{aligned}$$

where  $\text{Sh}(p, q-p)$  denotes the set of  $(p, q-p)$ -unshuffles, that is, a permutation  $\sigma \in S_q$  such that  $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$  and  $\sigma(p+1) < \sigma(p+2) < \cdots < \sigma(q)$ .

**Definition 2.3** (Contraction by a polyvector). For  $X \in \mathfrak{X}^p(A)$ , define the contraction operator

$$i_X: \Omega_A^q \longrightarrow \Omega_A^{q-p}$$

to be zero when  $q < p$ , and, when  $q \geq p$ , to be

$$i_X := (X \otimes \text{id}) \circ \Delta_{p,q-p}.$$

Equivalently,

$$\begin{aligned} i_X(a_0 da_1 \wedge \cdots \wedge da_q) &= a_0 \sum_{\sigma \in \text{Sh}(p,q-p)} \text{sgn}(\sigma) X(da_{\sigma(1)} \wedge \cdots \wedge da_{\sigma(p)}) \\ &\quad \cdot da_{\sigma(p+1)} \wedge \cdots \wedge da_{\sigma(q)}. \end{aligned}$$

Thus  $i_X$  is an  $A$ -linear operator of degree  $-p$  on  $\Omega_A^\bullet$ .

The map  $X \mapsto i_X$  is injective: its restriction to  $\Omega_A^p$  is precisely the functional  $X: \Omega_A^p \rightarrow A$ .

There is a second, equally basis-free contraction, with the roles of forms and polyvectors reversed. If  $\alpha \in \Omega_A^r$  and  $X \in \mathfrak{X}^p(A)$  with  $r \leq p$ , define

$$(i_\alpha X)(\beta) := X(\alpha \wedge \beta), \quad \beta \in \Omega_A^{p-r}. \quad (1)$$

Then  $i_\alpha X \in \mathfrak{X}^{p-r}(A)$ . In particular, if  $\alpha = da_1 \wedge \cdots \wedge da_r$ , then

$$i_\alpha X = i_{da_r} \cdots i_{da_1} X.$$

The type of the argument will always distinguish the two uses of  $i$ .

**2.2. The generalized Lie derivative and the Schouten bracket.** If  $D$  and  $E$  are homogeneous operators on  $\Omega_A^\bullet$ , of operator degrees  $|D|$  and  $|E|$ , respectively, their graded commutator is

$$[D, E]_{\text{gr}} := D \circ E - (-1)^{|D||E|} E \circ D.$$

**Definition 2.4** (Generalized Lie derivative). For  $X \in \mathfrak{X}^p(A)$ , define

$$L_X := [i_X, d]_{\text{gr}} = i_X d - (-1)^p d i_X.$$

It is an operator of degree  $1-p$  on  $\Omega_A^\bullet$ .

We now show that the operator identity suggested by Cartan calculus really does define a bracket on polyvector fields for an arbitrary  $A$ .

For  $X \in \mathfrak{X}^p(A)$  and  $Y \in \mathfrak{X}^q(A)$ , with  $p \geq 1$ , set

$$(X \circ Y)(a_1, \dots, a_{p+q-1}) := \sum_{\sigma \in \text{Sh}(q,p-1)} \text{sgn}(\sigma) X(Y(a_{\sigma(1)}, \dots, a_{\sigma(q)}), a_{\sigma(q+1)}, \dots, a_{\sigma(p+q-1)}). \quad (2)$$

When  $q = 0$ , the expression  $Y$  means the element  $Y \in A$ ; when  $p = 0$ , we set  $X \circ Y = 0$ .

**Proposition 2.5** (Operator characterization of the Schouten bracket). *For  $X \in \mathfrak{X}^p(A)$  and  $Y \in \mathfrak{X}^q(A)$  with  $p + q \geq 1$ , the alternating map*

$$[X, Y]_S := X \circ Y - (-1)^{(p-1)(q-1)} Y \circ X \quad (3)$$

*is a multiderivation, hence an element of  $\mathfrak{X}^{p+q-1}(A)$ , and it satisfies*

$$\boxed{i_{[X, Y]_S} = [L_X, i_Y]_{\text{gr}}}. \quad (4)$$

For  $p = q = 0$ , set  $[X, Y]_S = 0$ . Equation (4) uniquely determines  $[X, Y]_S$ .

*Proof.* The two Leibniz expansions in (3) have the same mixed terms with opposite signs; consequently their difference is a derivation in each argument. Alternation is built into the unshuffle sums. Thus (3) defines a member of  $\mathfrak{X}^{p+q-1}(A)$ .

To prove (4), apply both sides to a generator  $a_0 da_1 \wedge \cdots \wedge da_m$  of  $\Omega_A^m$ . If  $m \geq p + q - 1$ , expanding  $d$  and the two shuffle contractions gives

$$\begin{aligned} [L_X, i_Y]_{\text{gr}}(a_0 da_1 \wedge \cdots \wedge da_m) &= a_0 \sum_{\sigma \in \text{Sh}(p+q-1, m-p-q+1)} \text{sgn}(\sigma) \\ &\quad \cdot [X, Y]_S(a_{\sigma(1)}, \dots, a_{\sigma(p+q-1)}) \\ &\quad \cdot da_{\sigma(p+q)} \wedge \cdots \wedge da_{\sigma(m)}. \end{aligned}$$

The terms in which  $d$  acts on  $a_0$  cancel. The remaining terms are exactly  $i_{[X, Y]_S}$  by the definition of contraction. If  $m < p + q - 1$ , both sides vanish. Finally, uniqueness follows from the injectivity of  $Z \mapsto i_Z$ .  $\square$

In fact, we prefer to take (4) as the conceptual definition of the Schouten bracket, namely an element such that the contraction with it equals to the graded commutator of  $L_x$  and  $i_Y$ ; the formula (3) is indeed used to prove its existence without any smoothness or basis assumption.

**Proposition 2.6** (Cartan identities). *For homogeneous polyvectors  $X, Y, Z$ , one has*

$$[L_X, L_Y]_{\text{gr}} = L_{[X, Y]_S}, \quad (5)$$

$$[X, Y]_S = -(-1)^{(p-1)(q-1)} [Y, X]_S, \quad (6)$$

where  $p$  and  $q$  are the polyvector degrees of  $X$  and  $Y$ .

*Proof.* The graded Jacobi identity for graded commutators and  $d^2 = 0$  give  $[L_X, d]_{\text{gr}} = 0$ . Therefore

$$\begin{aligned} [L_X, L_Y]_{\text{gr}} &= [L_X, [i_Y, d]_{\text{gr}}]_{\text{gr}} \\ &= [[L_X, i_Y]_{\text{gr}}, d]_{\text{gr}} = [i_{[X, Y]_S}, d]_{\text{gr}} = L_{[X, Y]_S}. \end{aligned}$$

The shifted skew-symmetry is immediate from (3).  $\square$

The adjoint action on polyvectors is often written with the same Lie derivative notation,

$$L_X Y := [X, Y]_S.$$

Equation (5) expresses the compatibility of this action with the operator  $L_X$  on differential forms.

For later use, notice that if  $f \in A$  and  $X \in \mathfrak{X}^p(A)$ , then

$$[X, f]_S = i_{df} X. \quad (7)$$

In particular, for a vector field  $X \in \mathfrak{X}^1(A)$ ,  $[X, f]_S = X(f)$ .

**Corollary 2.7** (Mixed Cartan identity). *Let  $\pi \in \mathfrak{X}^2(A)$  and  $f \in A$ . With the convention*

$$L_P(Q) := [Q, P]_S,$$

*one has*

$$[L_\pi, i_{df}]_{\text{gr}} = L_{i_{df}\pi}.$$

*Equivalently,*

$$L_\pi i_{df} + i_{df} L_\pi = L_{i_{df}\pi}.$$

*Proof.* For  $Q \in \mathfrak{X}^q(A)$ , the graded Jacobi identity and (6) give

$$\begin{aligned} (L_\pi i_{df} + i_{df} L_\pi)(Q) &= [[Q, f]_S, \pi]_S + [[Q, \pi]_S, f]_S \\ &= [Q, [f, \pi]_S]_S. \end{aligned}$$

Since  $\deg f = 0$  and  $\deg \pi = 2$ , one has

$$[f, \pi]_S = [\pi, f]_S.$$

Moreover, by (7),

$$[\pi, f]_S = i_{df}\pi.$$

Consequently,

$$(L_\pi i_{df} + i_{df} L_\pi)(Q) = [Q, i_{df}\pi]_S = L_{i_{df}\pi}(Q).$$

□

**2.3. Poisson bivectors.** Let  $\pi \in \mathfrak{X}^2(A)$ . Define

$$\{f, g\}_\pi := \pi(df \wedge dg), \quad f, g \in A. \quad (8)$$

Because  $\pi$  is an alternating biderivation, this operation is skew-symmetric and satisfies the Leibniz rule in each argument. Its Hamiltonian derivation is

$$X_f := \{f, -\}_\pi = [\pi, f]_S = i_{df}\pi. \quad (9)$$

**Proposition 2.8** (Poisson criterion). *The bracket (8) is a Poisson bracket if and only if*

$$[\pi, \pi]_S = 0.$$

*More precisely, its Jacobiator is*

$$\{f, \{g, h\}_\pi\}_\pi + \{g, \{h, f\}_\pi\}_\pi + \{h, \{f, g\}_\pi\}_\pi = -\frac{1}{2}[\pi, \pi]_S(df \wedge dg \wedge dh). \quad (10)$$

*Proof.* The idea of the proof of the first statement is to apply the characterization given by (4) since  $[\pi, \pi]_S$  vanishes if and only if the contraction with this polyvector is 0. Namely we want to calculate  $i_{[\pi, \pi]_S}(df \wedge dg \wedge dh)$  to see when it vanishes. However since  $df \wedge dg \wedge dh$  is a 3-form while  $[\pi, \pi]_S$  is a 3-vector, the thing we need to calculate is exactly  $[\pi, \pi]_S(df \wedge dg \wedge dh)$  and this leads naturally to the second statement.

Now by (4),

$$\begin{aligned} i_{[\pi, \pi]_S}(df \wedge dg \wedge dh) &= [L_\pi, i_\pi]_{\text{gr}}(df \wedge dg \wedge dh) \\ &= ((i_\pi d - di_\pi)i_\pi - i_\pi(i_\pi d - di_\pi))(df \wedge dg \wedge dh). \end{aligned}$$

Now since  $d(df \wedge dg \wedge dh) = 0$ , and  $i_\pi^2$  decreases the degree by 4 so we also have  $i_\pi^2(df \wedge dg \wedge dh) = 0$ , the result above reduces to

$$2i_\pi di_\pi(df \wedge dg \wedge dh).$$

Expanding by the definition of  $i_\pi$  and comparing with the definition with the Jacobiator proves (10) and hence the proposition.  $\square$

### 3. DEL PEZZO SURFACES AND ISOLATED ELLIPTIC SINGULARITIES

#### 3.1. del Pezzo surfaces.

**Definition 3.1.** A *del Pezzo surface* is a smooth projective surface  $S$  that is obtained by blowing up  $\ell$  ( $0 \leq \ell \leq 8$ ) sufficiently general points in  $\mathbb{P}^2$  or  $\mathbb{P}^1 \times \mathbb{P}^1$ .

Let  $K_S$  denote the canonical bundle of  $S$ , and let  $K_S^{-1}$  be the anti-canonical bundle (the dual of  $K_S$ ); that is,

$$K_S = \Lambda^2 \Omega_S, \quad K_S^{-1} = \Lambda^2 T_S.$$

If we take

$$A = \Gamma(S, \Lambda^\bullet T_S)$$

and apply the discussion in Section 2, then a regular section

$$\pi \in \Gamma(S, K_S^{-1}) = \mathfrak{X}^2 A$$

gives rise to a Poisson structure on  $A$ , since  $[\pi, \pi]$  automatically vanishes in  $\mathfrak{X}^3 A = 0$ .

**Definition 3.2.** A regular section  $\pi \in \Gamma(S, K_S^{-1})$  is said to be *non-degenerate* if its divisor of zeroes is a reduced smooth curve.

**Proposition 3.3.** *When  $\pi$  is a non-degenerate section, the zero locus of  $\pi$  is an elliptic curve  $E \subset S$ , and  $X := S \setminus E$  is affine.*

*Proof.* The first statement is a simple application of the adjunction formula

$$K_E \cong K_S \otimes \mathcal{O}_S(E)|_E.$$

By definition of  $E$ ,  $\mathcal{O}_S(-E) \cong I_E$ , the ideal sheaf of  $E$ , and we consider the map  $\mathcal{O}_S \xrightarrow{\pi} K_S^{-1}$  given by the regular section  $\pi$ . After dualizing, this gives  $K_S \xrightarrow{\pi} \mathcal{O}_S$ , whose image is precisely the zero-locus ideal  $I_E$ . Thus  $K_S \cong I_E \cong \mathcal{O}_S(-E)$ , so  $K_E \cong \mathcal{O}_E$ . Hence the genus  $g$  of the reduced smooth curve  $E$  satisfies  $2g - 2 = 0$ , namely  $g = 1$ , and  $E$  is an elliptic curve.

The second statement follows from the two theorems quoted below. The first one is a well-known result from classical birational geometry. A proof using the birational classification of rational surfaces can be found in [Man86, Chapter IV, §24, Theorems 24.3–24.4]. The proof of the second theorem will be given below.  $\square$

**Theorem 3.4.** *A projective surface  $S$  is del Pezzo if and only if its anti-canonical bundle is ample.*

**Theorem 3.5.** *Suppose  $V$  is a projective variety and  $D$  is an effective ample divisor on  $V$ . Then  $V \setminus \text{Supp}(D)$  is an affine variety.*

*Proof.* Since  $D$  is ample, there exists an integer  $m > 0$  such that  $\mathcal{O}_V(mD)$  is very ample. Let

$$\iota : V \hookrightarrow \mathbb{P}^N$$

be the closed embedding defined by the complete linear system  $|mD|$ . If  $s_D$  denotes the canonical section of  $\mathcal{O}_V(D)$  whose zero divisor is  $D$ , then  $s_D^m \in H^0(V, \mathcal{O}_V(mD))$  has zero divisor  $mD$ . Hence there is a hyperplane  $H \subset \mathbb{P}^N$  such that

$$\iota^{-1}(H) = mD$$

scheme-theoretically. Since  $D$  and  $mD$  have the same support, we obtain

$$V \setminus \text{Supp}(D) = V \setminus \text{Supp}(mD) = \iota(V) \cap (\mathbb{P}^N \setminus H).$$

Now  $\mathbb{P}^N \setminus H \cong \mathbb{A}^N$ , and  $\iota(V) \cap (\mathbb{P}^N \setminus H)$  is closed in  $\mathbb{P}^N \setminus H$ . It is therefore affine.  $\square$

It is therefore reasonable to consider the affine del Pezzo surface  $X$  and its coordinate ring  $\mathbb{C}[X]$ . The latter has a natural algebraic symplectic structure given by

$$\pi^{-1} \in \Gamma(S \setminus E, K_S),$$

making  $\mathbb{C}[X]$  into a Poisson algebra; this is exactly the Poisson structure given by  $\pi$  after restricting to  $X \subset S$ . We will focus on this algebra in the sequel.

We denote the del Pezzo surface obtained by blowing up  $\ell$  points in sufficiently general position in  $\mathbb{P}^2$  by  $S_\ell$  (which has a more intrinsic definition). We focus on the cases  $\ell = 6, 7, 8$ . In these cases a special feature is that the affine del Pezzo surfaces

$$X_\ell := S_\ell \setminus E$$

can be embedded into  $\mathbb{C}^3$ , as summarized by the following theorem.

**Theorem 3.6.** *The following statements hold.*

- (i) *The affine del Pezzo surface  $X_\ell$  can be embedded into  $\mathbb{C}^3$  when  $\ell = 6, 7, 8$ .*
- (ii) *Its defining equation may be written in the form*

$$\phi_{p,q,r}^{\tau,\nu} := \tau xyz + P(x) + Q(y) + R(z) + \nu \in \mathbb{C}[x, y, z],$$

where

$$\begin{aligned} P(x) &= \frac{1}{p}x^p + \alpha_1 x^{p-1} + \cdots + \alpha_{p-1}x, \\ Q(y) &= \frac{1}{q}y^q + \beta_1 y^{q-1} + \cdots + \beta_{q-1}y, \\ R(z) &= \frac{1}{r}z^r + \gamma_1 z^{r-1} + \cdots + \gamma_{r-1}z, \end{aligned}$$

and  $\tau \in \mathbb{C}^\times$ ,  $\nu \in \mathbb{C}$ . The triples  $(p, q, r)$  are

$$(3, 3, 3), \quad (4, 4, 2), \quad (6, 3, 2)$$

in the cases  $\ell = 6, 7, 8$ , respectively.

We will omit the proof of these statements. Actually they can be deduced using [theorem 3.7](#) below, whose proof are also complicated and will be omitted in this survey.

**Theorem 3.7.** *The following statements hold.*

- (i) *For a projective elliptic curve  $E$ , its affine cone  $C_E$  has an isolated elliptic singularity at the cone vertex.*
- (ii) *For an affine del Pezzo surface  $X = S \setminus E$ , where  $S$  is the projective del Pezzo surface,  $X$  is a flat deformation of  $C_E$ .*

In view of this theorem, the “classification” of affine del Pezzo surfaces in [theorem 3.6](#) boils down to two parts: the classification of isolated elliptic singularities (at a cone vertex), and the classification of deformations of such singularities. The first part will be stated in [proposition 3.11](#) and the result we want for the second part is indeed saying that the deformation of isolated singularities on a hypersurface is semiuniversal, hence can be obtained by deforming the defining equation in some sense. The precise result is summarized as following.

**Theorem 3.8.** *The equations*

$$\phi_{p,q,r}^{\tau,\nu}(x, y, z) = 0$$

give a flat  $\mu$ -parameter family of affine del Pezzo surfaces of the corresponding type  $E_\ell$ , providing a semiuniversal deformation of the isolated elliptic singularity of the corresponding type  $E_\ell$ , where

$$\mu = p + q + r - 1, \quad \ell = 6, 7, 8.$$

For a proof for this theorem, one can refer to the standard construction of a semiuniversal deformation of an isolated hypersurface singularity from a basis of its Tjurina algebra; see [Loo84, Chapter 6, especially §6.C]. Since  $\phi^\tau$  is quasi-homogeneous, its Tjurina algebra agrees with its Jacobi ring. The relation with affine del Pezzo surfaces is explained in [EG10, Theorem 2.5.3(ii)]. The details will not be further explained.

**3.2. Quasi-homogeneous surface singularities.** We consider the coordinate ring  $\mathbb{C}[x, y, z]$  of  $\mathbb{C}^3$  as a nonnegatively graded algebra with homogeneous generators of degrees  $\deg x = a$ ,  $\deg y = b$ ,  $\deg z = c$ .

**Definition 3.9.** A polynomial  $\phi \in \mathbb{C}[x, y, z]$  is a (weighted-, equivalently, quasi-) homogeneous polynomial of weight  $\deg \phi = d$  if and only if one has  $\text{eu}(\phi) = d \cdot \phi$ , where

$$\text{eu} := ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + cz \frac{\partial}{\partial z}, \quad (11)$$

denotes the *Euler vector field*.

For any polynomial  $\phi \in \mathbb{C}[x, y, z]$  with an isolated singularity, its *Jacobi ring* is

$$\mathbb{J}(\phi) := \mathbb{C}[x, y, z] / \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).$$

If  $\phi$  is (weighted-) homogeneous of weight  $d$ , then  $0 \in \mathbb{C}^3$  is the only singular point. Furthermore, the Jacobi ring acquires a natural grading  $\mathbb{J}(\phi) = \bigoplus_{m \geq 0} \mathbb{J}^{(m)}(\phi)$ .

**Proposition 3.10.** *The Hilbert–Poincaré polynomial of this ring has the following formula:*

$$\sum_{m \geq 0} u^m \cdot \dim \mathbb{J}^{(m)}(\phi) = \frac{(u^{d-a} - 1)(u^{d-b} - 1)(u^{d-c} - 1)}{(u^a - 1)(u^b - 1)(u^c - 1)}. \quad (12)$$

*Proof.* Put  $S = \mathbb{C}[x, y, z]$ , graded by  $\deg x = a$ ,  $\deg y = b$ , and  $\deg z = c$ . Its Hilbert series is

$$H_S(u) = \frac{1}{(1 - u^a)(1 - u^b)(1 - u^c)}.$$

Since  $\phi$  has weighted degree  $d$ , its partial derivatives have degrees  $d - a$ ,  $d - b$ , and  $d - c$ . The singularity is isolated, so the Jacobian ideal

$$I = (\phi_x, \phi_y, \phi_z)$$

has finite codimension in  $S$ . Hence its three homogeneous generators form a regular sequence. Quotienting a graded ring by a homogeneous non-zero-divisor of degree  $e$  multiplies its Hilbert series by  $1 - u^e$ . Applying this three times gives

$$H_{\mathbb{J}(\phi)}(u) = H_S(u)(1 - u^{d-a})(1 - u^{d-b})(1 - u^{d-c}) = \frac{(1 - u^{d-a})(1 - u^{d-b})(1 - u^{d-c})}{(1 - u^a)(1 - u^b)(1 - u^c)}.$$

Multiplying both numerator and denominator by  $(-1)^3$  gives exactly (12).  $\square$

Set  $\mathcal{M}_\phi := \phi^{-1}(0) \subset \mathbb{C}^3$ . Specializing the RHS of (12) at  $u = 1$ , we get a formula

$$\dim \mathbb{J}(\phi) = \mu := \frac{(d-a)(d-b)(d-c)}{abc}, \quad (13)$$

for the *Milnor number* of the isolated singularity (at the origin) of the hypersurface  $\mathcal{M}_\phi$ .

**3.3. Simple elliptic singularities.** In this section we specialize to the case where the isolated singularity is elliptic. In particular, we want to obtain a kind of “classification” of such singularities, as in [proposition 3.11](#). To obtain this, we first pass to a projective curve.

Indeed, there is a general procedure for this. Let

$$\mathbb{P}^{a,b,c} := (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^\times$$

denote the weighted projective plane corresponding to the  $\mathbb{C}^\times$ -action with weights  $(a, b, c)$ , where  $\gcd(a, b, c) = 1$ . Restricting the projection  $\mathbb{C}^3 \setminus \{0\} \rightarrow \mathbb{P}^{a,b,c}$  to the punctured hypersurface, one obtains a map

$$\mathcal{M}_\phi \setminus \{0\} \longrightarrow \mathbb{P}(\mathcal{M}_\phi) \subset \mathbb{P}^{a,b,c}.$$

This way  $\mathcal{M}_\phi \setminus \{0\}$  becomes a principal  $\mathbb{C}^\times$ -bundle over  $\mathbb{P}(\mathcal{M}_\phi)$ , a projective curve. And indeed the type of the hypersurface  $\mathcal{M}_\phi$  is closely related to the integer

$$\varpi := d - a - b - c. \quad (14)$$

The case  $\varpi = 0$  corresponds to simple elliptic singularities  $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$  (for a reducible curve, all components must be rational). Specifically, one has the following classical result, cf. e.g. [B], and §6.1 below.

**Proposition 3.11.** *Let the variables  $x, y, z$  have degrees  $0 < a \leq b \leq c$ , such that  $\gcd(a, b, c) = 1$ .*

*(i) Let  $\phi \in \mathbb{C}[x, y, z]$  be an irreducible homogeneous polynomial of degree  $\deg \phi \leq a + b + c$ . Then, the projective curve  $\phi(x, y, z) = 0$  is either rational or elliptic.*

*(ii) Let  $d \leq a + b + c$  be such that, for a general homogeneous polynomial  $\phi$  of degree  $d$ , the projective curve  $\phi(x, y, z) = 0$  is elliptic. Then,  $d = a + b + c$ , and we have:*

- One of the following holds:

|                | $a$ | $b$ | $c$ | $d$ | $p := \frac{d}{a}$ | $q := \frac{d}{b}$ | $r := \frac{d}{c}$ | $\mu$ |
|----------------|-----|-----|-----|-----|--------------------|--------------------|--------------------|-------|
| 1. $E_6$ case: | 1   | 1   | 1   | 3   | 3                  | 3                  | 3                  | 8     |
| 2. $E_7$ case: | 1   | 1   | 2   | 4   | 4                  | 4                  | 2                  | 9     |
| 3. $E_8$ case: | 1   | 2   | 3   | 6   | 6                  | 3                  | 2                  | 10    |

(15)

*moreover, the integers  $(p-1, q-1, r-1)$  give the lengths of 3 legs of the corresponding extended Dynkin diagram of type  $\tilde{E}_6, \tilde{E}_7$ , or  $\tilde{E}_8$ .*

- *The homogeneous equation of the corresponding elliptic curve can be brought to the canonical form*

$$\phi^\tau(x, y, z) = \frac{x^p}{p} + \frac{y^q}{q} + \frac{z^r}{r} + \tau xyz = 0, \quad \text{where } \tau \in \mathbb{C}^\times. \quad (16)$$

We note in particular that, in the setting of, one has

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{a}{d} + \frac{b}{d} + \frac{c}{d} = \frac{a+b+c}{d} = 1.$$

And for the Milnor number, we have  $\dim \mathbb{J}(\phi^\tau) = \mu = \frac{(a+b)(a+c)(b+c)}{abc} = (\frac{d}{c} - 1)(\frac{d}{b} - 1)(\frac{d}{a} - 1) = (p-1)(q-1)(r-1)$ . The observation above reduces this formula to  $p + q + r - 1$ , agreeing with the  $\mu$  given in [theorem 3.8](#).

## 4. POISSON COHOMOLOGY

In this section we will focus on the Poisson structure on a hypersurface embedded in  $\mathbb{C}^3$  with an isolated singularity. To introduce this Poisson structure, we first consider a Poisson structure on  $\mathbb{C}^3$ , namely a Poisson bracket given by  $\mathbb{C}[x, y, z]$ . For any  $\phi \in \mathbb{C}[x, y, z]$ , it is explicitly given by the following formulas:

$$\{x, y\} = \frac{\partial \phi}{\partial z}, \quad \{y, z\} = \frac{\partial \phi}{\partial x}, \quad \{z, x\} = \frac{\partial \phi}{\partial y}.$$

In view of [section 2](#), this Poisson bracket is determined by a bivector  $\pi \in \mathfrak{X}^2(\mathbb{C}^3)$  such that  $i_\pi \text{vol} = d\phi$ , where  $\text{vol}$  is the nowhere vanishing volume form  $dx \wedge dy \wedge dz$ . And this Poisson bracket is determined by the equation

$$\{f, g\} \cdot \text{vol} = d\phi \wedge df \wedge dg, \quad \forall f, g \in \mathbb{C}[x, y, z].$$

We denote this Poisson algebra structure on  $\mathbb{C}[x, y, z]$  by  $\mathcal{A}_\phi$ . By the characterization above one may easily see that  $\phi$  is in the center of  $\mathcal{A}_\phi$  (i.e.  $\{\phi, f\} = 0$  for all  $f \in \mathcal{A}_\phi$ ). (This can also be checked by direct calculation.) Hence we can consider the quotient Poisson algebra  $\mathcal{B}_\phi := \mathcal{A}_\phi / (\phi)$ .

The main theorem we want to explain in this section is the following Poisson enhancement of [theorem 3.8](#).

**Theorem 4.1.** *The family of Poisson algebras  $\{\mathcal{B}_\phi, \phi = c \cdot \phi_{P,Q,R}^{\tau,\nu}, c \in \mathbb{C}^\times\}$  provides a semiuniversal Poisson deformation of  $\mathcal{B}_{\phi^\tau}$ .*

As explained in [section 1.1](#), to prove a deformation is semiuniversal for Poisson structure we need to know about the Poisson cohomology. So in this chapter we first calculate the Poisson cohomology of  $B_\phi$ , using several results of Pichereau concerning the Poisson cohomology of the total space  $A_\phi$  that she calculated in [\[Pic06\]](#), where  $\phi$  is a general polynomial in  $\mathbb{C}[x, y, z]$ . Finally, by identifying the corresponding Kodaira-Spencer map with an explicit form, we use the results to prove [theorem 4.1](#).

However, the general definition of Poisson cohomology of a possibly non-smooth algebra involves cotangent complex and that would be omitted. We will reduce to the case of smooth (complex of) algebras where the Poisson cohomology can be shown to be given by the LKB complex, that is,  $PH^\bullet(A) = H^\bullet(\mathfrak{X}^\bullet(A), L_\pi)$ ,

where the action of  $L_\pi$  is given by the Schouten bracket, i.e.  $L_\pi Y = [\pi, Y]_S$ .

**4.1. Poisson cohomology of a hypersurface.** In view of Pichereau's results concerning the Poisson cohomology of the total space  $A_\phi$ , listed in the next subsection, we need to know the Poisson cohomology of a hypersurface, namely an algebra of the form  $B_\phi := \mathbb{C}[\mathcal{M}]/(\phi)$ , where  $\mathcal{M}$  is a smooth Poisson variety and  $\phi$  is a regular function in the Poisson center. We first note the following lemma concerning the anti-commutativity of  $L_\pi$  and  $i_{d\phi}$  since these differentials are differentials of a double complex in the proof of the main result of this subsection.

**Lemma 4.2.** *For a central function  $\phi$ , the operators  $L_\pi$  and  $i_{d\phi}$  anti-commute:*

*Proof.* By [corollary 2.7](#),  $L_\pi i_{d\phi} + i_{d\phi} L_\pi = L_{i_{d\phi}\pi}$ . Since  $\phi$  is central, (9) gives  $i_{d\phi}\pi = [\pi, \phi]_S = X_\phi = 0$ . Hence we get the result.  $\square$

We can now state the main result in this subsection.

**Proposition 4.3.** *Let  $\phi \in Z(\mathbb{C}[\mathcal{M}])$  be a central regular function on  $\mathcal{M}$ . Assume that  $\phi$  has only isolated singularities and that there exists a vector field  $\text{eu} \in \mathfrak{X}^1(\mathcal{M})$  such that*

$$\text{eu}(\phi) = c\phi, \quad c \neq 0.$$

Then, for the Poisson cohomology of the algebra  $B_\phi := \mathbb{C}[\mathcal{M}]/(\phi)$ , there is a convergent first spectral sequence

$$E_1^{p,q} \implies \mathrm{gr}_F^p PH^{p+q}(B_\phi),$$

whose  $E_1$ -term has the following form:

$$\begin{array}{ccccc}
 \dots & \dots & \dots & \dots & \dots \\
 \uparrow L_\pi & \uparrow L_\pi & \uparrow L_\pi & \uparrow L_\pi & \uparrow L_\pi \\
 \mathfrak{X}^4(B_\phi) & 0 & 0 & \mathbb{J}(\phi)\mathrm{eu} & \mathbb{J}(\phi) \\
 \uparrow L_\pi & \uparrow L_\pi & \uparrow L_\pi & \uparrow L_\pi & \\
 \mathfrak{X}^3(B_\phi) & 0 & \mathbb{J}(\phi)\mathrm{eu} & \mathbb{J}(\phi) & \\
 \uparrow L_\pi & \uparrow L_\pi & \uparrow L_\pi & & \\
 \mathfrak{X}^2(B_\phi) & \mathbb{J}(\phi)\mathrm{eu} & \mathbb{J}(\phi) & & \\
 \uparrow L_\pi & \uparrow L_\pi & & & \\
 \mathfrak{X}^1(B_\phi) & \mathbb{J}(\phi) & & & \\
 \uparrow L_\pi & & & & \\
 \mathfrak{X}^0(B_\phi) & & & & 
 \end{array} \tag{17}$$

Here the leftmost column is the LKB complex of the Poisson algebra  $B_\phi$ , and  $\mathbb{J}(\phi)$  denotes the Jacobi ring of  $\phi$ , as in [section 3.2](#).

*Proof.* The idea of the proof is to provide a dg-algebra resolution of the non-smooth algebra  $B_\phi$ . Set

$$A := \mathbb{C}[\mathcal{M}], \quad DA := A \otimes \mathbb{C}[\tau]/(\tau^2),$$

a graded super-commutative algebra in which  $A$  is the even part placed in degree 0 and  $\tau$  is an odd generator of degree  $-1$ . Introduce a differential

$$\nabla : DA^\bullet \longrightarrow DA^{\bullet+1}$$

defined as the odd super-derivation satisfying

$$\nabla(A) = 0, \quad \nabla(\tau) = \phi.$$

One can also write it out explicitly as  $\nabla = \phi \frac{\partial}{\partial \tau}$ . As a complex this is the two-term complex  $A \xrightarrow{\phi} A$ . Thus  $H_0(DA) = B_\phi$  and  $H_i(DA) = 0$  for  $i \neq 0$ .

We can also extend the Poisson bracket to  $DA$  by  $\tau$ -linearity, making  $DA$  a Poisson dg algebra quasi-isomorphic to  $B_\phi$ . Hence

$$PH^\bullet(B_\phi) \cong PH^\bullet(DA).$$

The polyvector fields on  $DA$  are generated by those on  $\mathcal{M}$  and the derivation with respect to the odd variable  $\tau$ :

$$\mathfrak{X}^\bullet(DA) \cong \mathfrak{X}^\bullet(\mathcal{M}) \otimes_{\mathbb{C}} \mathbb{C} \left[ \tau, \frac{\partial}{\partial \tau} \right] / (\tau^2).$$

Now For the LKB complex on a dg algebra, the differential is  $\tilde{\nabla} + L_\pi$ , where  $\tilde{\nabla}$  is induced by  $\nabla$ . By definition,

$$\tilde{\nabla}(P) = [\nabla, P].$$

Using the identity

$$[fX, P] = f[X, P] - (i_{df}P) \wedge X$$

with  $f = \phi$  and  $X = \partial/\partial\tau$ , one finds

$$[\nabla, P] = -(i_{d\phi}P) \wedge \frac{\partial}{\partial\tau}.$$

Indeed  $[\partial/\partial\tau, P] = 0$  when either  $P = \partial/\partial\tau$  or  $P \in \mathfrak{X}^\bullet(A)$ .

One may therefore write the LKB complex in the form (where we denote  $\frac{\partial}{\partial\tau}$  as  $\partial_\tau$  for brevity).

$$\left( \mathfrak{X}^\bullet(\mathcal{M}) \otimes \frac{\mathbb{C}[\tau, \partial_\tau]}{(\tau^2)}, \phi\partial_\tau + \tau i_{d\phi} + L_\pi \right).$$

Let

$$T_\phi := T_{\mathcal{M}}|_{\phi^{-1}(0)}, \quad \Lambda_\phi^\bullet := \Gamma(\phi^{-1}(0), \Lambda^\bullet T_\phi) \cong B_\phi \otimes_A \mathfrak{X}^\bullet(\mathcal{M}).$$

There is a natural algebra projection

$$\text{pr} : \mathfrak{X}^\bullet(\mathcal{M}) \otimes \frac{\mathbb{C}[\tau, \partial_\tau]}{(\tau^2)} \longrightarrow \Lambda_\phi^\bullet \otimes \mathbb{C}[\partial_\tau],$$

We will explain this projection map in detail below. Let

$$\Lambda_\phi := \Gamma(\phi^{-1}(0), \Lambda^\bullet T_\phi) \cong B_\phi \otimes_A \mathfrak{X}^\bullet(\mathcal{M}).$$

The projection above is explicitly given on simple tensors by

$$x \otimes P(\partial_\tau) + \tau y \otimes Q(\partial_\tau) \longmapsto [\bar{x}] \otimes P(\partial_\tau),$$

where  $\bar{x}$  denotes the residue class of  $x$  in  $\Lambda_\phi \cong \mathfrak{X}^\bullet(\mathcal{M})/\phi\mathfrak{X}^\bullet(\mathcal{M})$ . It is a projection of dg algebras.

We now want to show that the projection is a quasi-isomorphism. Its kernel  $K$  fits into the short exact sequence

$$0 \longrightarrow K \longrightarrow C^\bullet \xrightarrow{\text{pr}} \Lambda_\phi^\bullet \longrightarrow 0.$$

Thus it is enough to prove that  $K$  is acyclic.

Write  $K = K^0 \oplus K^1$ , corresponding respectively to terms without and with a factor  $\tau$ . Decompose the total differential as  $D = d_0 + d_1$ , where

$$d_0 = \phi\partial_\tau : K^1 \longrightarrow K^0, \quad d_1 = \tau i_{d\phi} + L_\pi.$$

By the preceding lemma,  $d_1^2 = 0$ , and

$$d_1 d_0 + d_0 d_1 = 0.$$

Multiplication by  $\phi$  is injective, so  $d_0$  is injective; by its definition on the kernel it is also surjective, hence an isomorphism.

Suppose  $Dz = 0$  and write  $z = z_0 + z_1$ , where  $z_i \in K^i$ . Then

$$d_1 z_1 = 0, \quad d_1 z_0 + d_0 z_1 = 0.$$

Using the inverse of  $d_0$  and the anti-commutation relation, one constructs  $w = d_0^{-1}(z_0)$  with  $Dw = z$ . Hence every cocycle in  $K$  is a coboundary, and  $\text{pr}$  is a quasi-isomorphism.

Consequently the Poisson cohomology of  $B_\phi$  may be computed as the hypercohomology of the dg algebra represented by the following mixed complex:

$$\begin{array}{ccccc}
 \cdots & & \cdots & & \cdots \\
 \uparrow L_\pi & & \uparrow L_\pi & & \uparrow L_\pi \\
 \Lambda_\phi^2 & \xrightarrow{i_{d\phi}} & \Lambda_\phi^1 & \xrightarrow{i_{d\phi}} & \Lambda_\phi^0 \\
 \uparrow L_\pi & & \uparrow L_\pi & & \\
 \Lambda_\phi^1 & \xrightarrow{i_{d\phi}} & \Lambda_\phi^0 & & \\
 \uparrow L_\pi & & & & \\
 \Lambda_\phi^0 & & & & 
 \end{array}$$

We regard the mixed complex displayed above as a first-quadrant bicomplex  $K$ , whose horizontal and vertical differentials are  $i_{d\phi}$  and  $L_\pi$ , respectively. The associated spectral sequence satisfies

$$E_1^{p,q} = H^{p,q}(K, i_{d\phi}),$$

and its first differential  $d_1$  is induced by  $L_\pi$ . This is well defined because  $L_\pi$  anti-commutes with  $i_{d\phi}$ .

For each  $q \geq 0$ , let

$$\Lambda_\phi^{(q)} : \Lambda_\phi^q \xrightarrow{i_{d\phi}} \Lambda_\phi^{q-1} \xrightarrow{i_{d\phi}} \cdots \xrightarrow{i_{d\phi}} \Lambda_\phi^1 \xrightarrow{i_{d\phi}} \Lambda_\phi^0$$

denote the complex appearing in the  $q$ -th row, where  $\Lambda_\phi^j$  is placed in degree  $j$ . We compute its cohomology in the following claim.

*Claim.* For every  $q \geq 0$ , one has

$$H^q(\Lambda_\phi^{(q)}, i_{d\phi}) \cong \mathfrak{X}^q(B_\phi).$$

Moreover,

$$H^j(\Lambda_\phi^{(q)}, i_{d\phi}) = 0 \quad \text{for } j \notin \{0, 1, q\},$$

and

$$H^0(\Lambda_\phi^{(q)}, i_{d\phi}) \cong \mathbb{J}(\phi).$$

If  $q > 1$ , then also

$$H^1(\Lambda_\phi^{(q)}, i_{d\phi}) \cong \mathbb{J}(\phi)\text{eu}.$$

*Proof of the claim.* Put

$$A = \mathbb{C}[\mathcal{M}], \quad B_\phi = A/(\phi), \quad \mathcal{M}_\phi = \phi^{-1}(0).$$

We first prove the assertion concerning the leftmost cohomology. The module of Kähler differentials of  $B_\phi$  has the presentation

$$B_\phi \cdot d\phi \longrightarrow \Omega_A^1 \otimes_A B_\phi \longrightarrow \Omega_{B_\phi}^1 \longrightarrow 0.$$

A  $q$ -polyderivation of  $B_\phi$  is equivalently a  $B_\phi$ -linear alternating map

$$\bigwedge^q \Omega_{B_\phi}^1 \longrightarrow B_\phi.$$

Pulling such a map back to  $\bigwedge^q(\Omega_A^1 \otimes_A B_\phi)$  gives a section  $s \in \Lambda_\phi^q$  which vanishes whenever one of its arguments is  $d\phi$ . This condition is precisely

$$i_{d\phi}s = 0.$$

Conversely, every section  $s \in \Lambda_\phi^q$  satisfying  $i_{d\phi}s = 0$  descends through the above presentation to a  $q$ -polyderivation of  $B_\phi$ . Therefore

$$\mathfrak{X}^q(B_\phi) \cong \ker(i_{d\phi}: \Lambda_\phi^q \longrightarrow \Lambda_\phi^{q-1}).$$

Since  $\Lambda_\phi^q$  is the leftmost term of  $\Lambda_\phi^{(q)}$ , there is no incoming differential, and hence

$$H^q(\Lambda_\phi^{(q)}, i_{d\phi}) \cong \mathfrak{X}^q(B_\phi).$$

It remains to compute the other cohomology groups. Consider the truncated Koszul complex

$$\mathfrak{X}^{(q)}: \mathfrak{X}^q(\mathcal{M}) \xrightarrow{i_{d\phi}} \mathfrak{X}^{q-1}(\mathcal{M}) \xrightarrow{i_{d\phi}} \cdots \xrightarrow{i_{d\phi}} \mathfrak{X}^0(\mathcal{M}).$$

Locally, this is the Koszul complex associated with the partial derivatives of  $\phi$ . Since  $\phi$  has only isolated singularities, these partial derivatives form a regular sequence. Consequently, the full Koszul complex is exact in positive degrees, and its degree-zero cohomology is

$$H^0(\mathfrak{X}^{(q)}, i_{d\phi}) = \frac{A}{(\partial\phi)} = \mathbb{J}(\phi).$$

After truncating at degree  $q$ , exactness can fail only at the two ends of the complex. Hence  $\mathfrak{X}^{(q)}$  is exact except possibly at its leftmost and rightmost terms.

Since

$$\Lambda_\phi^j \cong \mathfrak{X}^j(\mathcal{M})/\phi\mathfrak{X}^j(\mathcal{M}),$$

multiplication by  $\phi$  gives a short exact sequence of complexes

$$0 \longrightarrow \mathfrak{X}^{(q)} \xrightarrow{\cdot\phi} \mathfrak{X}^{(q)} \longrightarrow \Lambda_\phi^{(q)} \longrightarrow 0.$$

The corresponding long exact sequence immediately gives

$$H^j(\Lambda_\phi^{(q)}, i_{d\phi}) = 0 \quad \text{for } j \notin \{0, 1, q\}.$$

For  $q > 1$ , its final part is

$$0 \longrightarrow H^1(\Lambda_\phi^{(q)}, i_{d\phi}) \longrightarrow \mathbb{J}(\phi) \xrightarrow{\cdot\phi} \mathbb{J}(\phi) \longrightarrow H^0(\Lambda_\phi^{(q)}, i_{d\phi}) \longrightarrow 0.$$

Indeed, the two copies of  $\mathbb{J}(\phi)$  here are the degree-zero cohomology groups of the two Koszul complexes in the short exact sequence.

By assumption,

$$i_{d\phi}\text{eu} = \text{eu}(\phi) = c\phi, \quad c \neq 0.$$

Thus

$$\phi = c^{-1}i_{d\phi}\text{eu}$$

belongs to the Jacobi ideal, so its class in  $\mathbb{J}(\phi)$  vanishes. Hence multiplication by  $\phi$  on  $\mathbb{J}(\phi)$  is the zero map. Exactness therefore yields

$$H^0(\Lambda_\phi^{(q)}, i_{d\phi}) \cong \mathbb{J}(\phi), \quad H^1(\Lambda_\phi^{(q)}, i_{d\phi}) \cong \mathbb{J}(\phi).$$

More precisely, the class of  $c^{-1}\text{eu}$  is mapped to  $1 \in \mathbb{J}(\phi)$  by the connecting morphism. Up to the nonzero scalar  $c$ , this identifies the latter group with

$$\mathbb{J}(\phi)\text{eu}.$$

This proves the claim.

Applying the claim to every row of  $K$  gives exactly the  $E_1$ -page displayed in (17). The differential  $d_1$  is induced by the remaining differential  $L_\pi$ . Since  $K$  is a first-quadrant bicomplex, its standard filtration gives a convergent spectral sequence

$$E_1^{p,q} \implies \mathrm{gr}_F^p PH^{p+q}(B_\phi),$$

as required.  $\square$

**4.2. Pichereau's results.** We now specialize to the setting of the quasi-homogeneous case. Thus, let

$$0 < a \leq b \leq c$$

be positive integers such that  $\gcd(a, b, c) = 1$ . Let  $\mathrm{eu}$  denote the corresponding Euler vector field on  $M = \mathbb{C}^3$ , and let  $\Upsilon$  be the standard three-vector  $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$  which is dual to the volume form  $dx \wedge dy \wedge dz$ .

Given a homogeneous polynomial  $\phi \in \mathbb{C}[M] = \mathbb{C}[x, y, z]$ , we put  $\pi := i_{d\phi}\Upsilon$ , and let  $\mathcal{A}_\phi$  denote the corresponding Poisson algebra.

Pichereau computed the Poisson cohomology groups of  $\mathcal{A}_\phi$  explicitly [Pic06, Propositions 4.2, 4.5, 4.7, and 4.8]. To state the result, set  $\mu := \dim_{\mathbb{C}} \mathbb{J}(\phi)$ , to be the Milnor number and choose homogeneous elements

$$1, f_1, \dots, f_{\mu-1} \in \mathbb{C}[x, y, z]$$

whose residue classes modulo the Jacobi ideal form a  $\mathbb{C}$ -basis of the Jacobi ring  $\mathbb{J}(\phi)$ . We regard

$$i_{df_k}\Upsilon \in \mathfrak{X}^2(\mathbb{C}^3), \quad k = 1, \dots, \mu - 1,$$

as elements of the LKB complex of  $\mathcal{A}_\phi$ .

**Proposition 4.4** (Pichereau). *Let  $\phi$  be a homogeneous polynomial with an isolated singularity and suppose that*

$$\deg \phi = a + b + c.$$

*Then the Poisson cohomology of  $\mathcal{A}_\phi$  vanishes in degrees greater than or equal to 4. Moreover, the following statements hold.*

(i) *One has*

$$PH^0(\mathcal{A}_\phi) = \mathbb{C}[\phi].$$

*Furthermore,*

$$PH^1(\mathcal{A}_\phi) = \mathbb{C}[\phi]\mathrm{eu}$$

*is a free  $\mathbb{C}[\phi]$ -module of rank one, generated by  $\mathrm{eu}$ .*

(ii) *The group  $PH^3(\mathcal{A}_\phi)$  is a free  $\mathbb{C}[\phi]$ -module of rank  $\mu$ , with basis*

$$\Upsilon, f_1\Upsilon, \dots, f_{\mu-1}\Upsilon.$$

*Similarly,  $PH^2(\mathcal{A}_\phi)$  is a free  $\mathbb{C}[\phi]$ -module with basis*

$$i_{df_1}\Upsilon, \dots, i_{df_{\mu-1}}\Upsilon, \pi.$$

Pichereau also computed the cohomology groups of the LKB complex of the singular Poisson algebra

$$B_\phi := \mathbb{C}[x, y, z]/(\phi)$$

associated with a quasi-homogeneous polynomial  $\phi \in \mathbb{C}[x, y, z]$  of arbitrary positive weight  $d$ . More precisely,

$$\mathfrak{X}^p(B_\phi) = 0 \quad \text{for all } p > 2,$$

and the cohomology of the LKB differential  $L_\pi$  is [Pic06, Propositions 5.3, 5.5, and 5.6]

$$\begin{aligned} H^0(\mathfrak{X}^\bullet(B_\phi), L_\pi) &= \mathbb{C}, \\ H^1(\mathfrak{X}^\bullet(B_\phi), L_\pi) &= \mathbb{J}^{(\varpi)}(\phi) \text{ eu}, \\ H^2(\mathfrak{X}^\bullet(B_\phi), L_\pi) &\cong \mathbb{J}^{(\varpi)}(\phi) \pi. \end{aligned} \tag{18}$$

Here

$$\varpi := d - a - b - c,$$

and  $\mathbb{J}^{(\varpi)}(\phi)$  denotes the homogeneous component of weight  $\varpi$  in the Jacobi ring. Equivalently, with the notation used above,

$$\mathbb{J}^{(\varpi)}(\phi) = \text{span}_{\mathbb{C}} \{ \overline{f_j} \mid \deg f_j = \varpi \}.$$

**4.3. Poisson cohomology of  $B_\phi$  and proof of theorem 4.1.** With the results of Pichereau and the preceding proposition at hand, we can calculate the Poisson cohomology of  $B_\phi$ . For a proof of this, one can see [?, Section 5.5] and the proof therein which is not so hard with all the results we have above.

**Theorem 4.5.** *The Poisson cohomology of  $B_\phi$  is as follows:*

$$\begin{aligned} PH^0(B_\phi) &= \mathbb{C}, \\ PH^1(B_\phi) &= \mathbb{J}^{(\varpi)}(\phi), \\ PH^2(B_\phi) &= \mathbb{J}^{(\varpi)}(\phi) \oplus \mathbb{J}(\phi), \\ PH^k(B_\phi) &= \mathbb{J}(\phi) \quad \text{for } k \geq 3, \end{aligned}$$

where

$$\varpi := d - (a + b + c),$$

and the Jacobi ring  $\mathbb{J}(\phi)$  is as in section 3.2.

We now use the theorem to prove theorem 4.1.

*Proof of theorem 4.1.* Let

$$S = \mathbb{C}^2 \times S_p \times S_q \times S_r \times \mathbb{C}^\times,$$

where the parameters  $\tau, \nu$  form coordinates in the first factor  $\mathbb{C}^2$ , the affine linear spaces  $S_p, S_q, S_r$  correspond to the coefficients  $\alpha_i$  of  $P$ ,  $\beta_i$  of  $Q$ , and  $\gamma_i$  of  $R$ , and the parameter  $c$  is a coordinate on the last factor  $\mathbb{C}^\times$ . This space parametrizes the family

$$\{ B_\phi \mid \phi = c \phi_{p,q,r}^{\tau,\nu}, \ c \in \mathbb{C}^\times \}.$$

Let  $o \in S$  be the “origin”, namely the point where  $c = 1$  and all other parameters vanish; there  $\phi_{p,q,r}^{\tau,\nu}$  reduces to  $\phi^\tau$ .

By definition, we need to show that the Kodaira-Spencer map

$$T_o S \longrightarrow PH^2(B_{\phi^\tau})$$

is an isomorphism. By the preceding theorem,

$$PH^2(B_{\phi^\tau}) \cong \mathbb{J}(\phi^\tau) \oplus \mathbb{C},$$

where the second summand is the one-dimensional contribution recorded in the source. Moreover,

$$\dim T_o S = (p-1) + (q-1) + (r-1) + 2 = \mu + 1 = \dim PH^2(B_{\phi^\tau}),$$

so it remains to prove that the Kodaira-Spencer map is injective.

Explicitly, a tangent vector to  $S$  is represented by a polynomial

$$\dot{F} = \dot{c}\phi^\tau + \dot{\tau}xyz + \dot{P} + \dot{Q} + \dot{R} + \dot{\nu}.$$

Equivalently, introducing a formal variable  $\varepsilon$  with  $\varepsilon^2 = 0$ , one has  $\dot{F} = \frac{F_\varepsilon - F_0}{\varepsilon}$ , where

$$\begin{aligned} F_0 &= c\phi^\tau, \\ F_\varepsilon &= (c + \varepsilon\dot{c}) \left( \phi^\tau + \varepsilon(\dot{\tau}xyz + \dot{P} + \dot{Q} + \dot{R} + \dot{\nu}) \right). \end{aligned}$$

Writing  $\dot{F} = \dot{c}\phi^\tau + \dot{F}'$ , in this case the Kodaira-Spencer map is the map that sends it to

$$([\dot{F}'], \dot{c}) \in \mathbb{J}(\phi^\tau) \oplus \mathbb{C}.$$

Hence it remains to show that if  $\dot{c} = 0$  and  $\dot{F}'$  belongs to the Jacobi ideal, then  $\dot{F}' = 0$ . Put

$$f_x = x^{p-1} + \tau yz, \quad f_y = y^{q-1} + \tau xz, \quad f_z = z^{r-1} + \tau xy.$$

Suppose

$$\dot{F}' = \dot{\nu} + \sum_{i=1}^{p-1} a_i x^i + \sum_{j=1}^{q-1} b_j y^j + \sum_{k=1}^{r-1} c_k z^k + \dot{\tau}xyz = Af_x + Bf_y + Cf_z,$$

where  $A, B, C$  are polynomials.

Set  $y = z = 0$ . Then

$$x^{p-1}A(x, 0, 0) = \dot{\nu} + \sum_{i=1}^{p-1} a_i x^i.$$

Comparison of degrees shows immediately that  $\dot{\nu} = 0$  and all  $a_i$  must vanish except possibly  $a_{p-1}$ . Similar arguments with the other axes show that all  $b_j, c_k$  vanish except possibly  $b_{q-1}, c_{r-1}$ . Thus the left-hand side is reduced to

$$a_{p-1}x^{p-1} + b_{q-1}y^{q-1} + c_{r-1}z^{r-1} + \dot{\tau}xyz.$$

The only pure monomial in  $x$  on  $Af_x + Bf_y + Cf_z$  comes from  $x^{p-1}A$ , forcing  $A = a_{p-1}$  to be constant. But this introduces a term  $a_{p-1}\tau yz$ , which cannot occur on the left-hand side. Hence  $a_{p-1} = 0$ , and similarly

$$b_{q-1} = c_{r-1} = 0.$$

It remains only to show that  $\dot{\tau} = 0$ .

One can prove this by a direct discussion in all three cases. A more elegant way is to give  $x, y, z$  fractional weights  $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}$ , respectively. Then  $xyz$  has weight 1, while  $f_x, f_y, f_z$  have weights  $1 - \frac{1}{p}, 1 - \frac{1}{q}, 1 - \frac{1}{r}$ . Thus  $A, B, C$  would have to have weights  $1/p, 1/q, 1/r$ , respectively. Starting discussion from this point may be more efficient since bounds their possible degrees and gives  $\dot{\tau} = 0$ . The remaining case-by-case details are omitted.  $\square$

#### ACKNOWLEDGMENTS

I'm much obliged to my mentor Prof. Victor Ginzburg for introducing me such an interesting topic and helping me figure out problems that I met during the reading with his patient guidance. I have gained so much insight from him during this summer. Besides, I would like to thank Prof. Peter May for organizing the mathematics REU program at the University of Chicago and providing such a wonderful research opportunity. Finally, my special thanks go to my dearest friends Shile Wang and Wuhan Zhou, for their accompany during our trip in Chicago city and the joy and happiness they bring to me.

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