

Basic Properties of Quantum Schubert Polynomials

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December 26, 2024

Abstract

In this paper, we will go through the paper of Fomin, Gelfand and Postnikov 1997, especially focusing on exhibiting the basic algebraic properties of quantum Schubert polynomials, mostly by working on pure algebraic computation of operators. Finally we will show that these polynomials are the representatives for the quantum cohomology ring using the orthogonal relation.

1 Introduction

Consider the flag manifold Fl_n of complete flags in the n -dimensional complex linear space $\mathbb{C}^n$, it's well-known that its cohomology ring has a presentation below, where I_n is generated by the symmetric polynomials in variables $x_1, \dots, x_n$ without constant term:

$$H^*(Fl_n, \mathbb{Z}) \cong \mathbb{Z}[x_1, \dots, x_n]/I_n. \quad (1)$$

Viewing this cohomology ring in another way, we shall decompose it into Schubert cells. The corresponding cohomology classes σ_w to the even-dimensional cells, indexed by permutations $w \in S_n$ form a basis of $H^*(Fl_n, \mathbb{Z})$. We've shown that the coset representatives of σ_w under the isomorphism above are exactly the Schubert polynomials $\mathfrak{S}_w$.

The new questions arise when the quantumized version is taken into consideration. With motivation from physics, we define the quantum cohomology ring $QH^*(Fl_n, \mathbb{Z})$ and it is in fact canonically isomorphic to the tensor product $H^*(Fl_n, \mathbb{Z}) \otimes \mathbb{Z}[q_1, \dots, q_{n-1}]$ as an Abelian group, where q_i are formal variables. The additive structure of this ring is essentially the same as the original cohomology ring, but the multiplicative nature will be quite different.

It is proved earlier that we have the following canonical ring isomorphism:

$$\mathrm{QH}^*(Fl_n, \mathbb{Z}) \cong \mathbb{Z}[q_1, \dots, q_{n-1}][x_1, \dots, x_n]/I_n^q, \quad (2)$$

where I_n^q are generated by the quantum elementary polynomials, defined in section 2, which specialize to the ordinary elementary symmetric polynomials when $q_i = 0$. We shall denote $\mathbb{Z}[q_1, \dots, q_{n-1}][x_1, \dots, x_n] = \mathbb{Z}[q][x]$ from now on. To understand the multiplicative structure of the quantum cohomology ring, a way is to recover the similar works we've finished for the ordinary cohomology ring: to find the combinatorial representatives under the isomorphism and do the jobs in some combinatorial model. In view of this, the key aim of the paper, is to find the elements in $\mathbb{Z}[q][x]/I_n^q$ that is the image of the Schubert classes in $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ under the isomorphism above.

In fact, since we've already know the isomorphism $H^*(Fl_n, \mathbb{Z}) \cong \mathbb{Z}[x_1, \dots, x_n]/I_n$, tensor this with $\mathbb{Z}[q]$ will ensure the existence of a $\mathbb{Z}[q]$ -linear map from $\mathbb{Z}[q, x]/I_n$ to $\mathbb{Z}[q, x]/I_n^q$. So it is even possible to do the work completely out of the geometric setting. Actually, in the final proof of the main theorem, we only used very little of the geometric property. To state more explicitly how we can proceed in an algebraic or combinatorial manner, we are looking for a $\mathbb{Z}[q]$ -linear map sending a certain basis of $\mathbb{Z}[q, x]/I_n$ to another basis of $\mathbb{Z}[q, x]/I_n^q$ that realize the quantization process, and so will be called the quantization map. Then with notations of standard elementary monomials to be introduced in section 2, we can state the purely algebraic version of the theorem:

Theorem 1.1. The quantization map is the $\mathbb{Z}[q]$ -linear map $\mathbb{Z}[q, x]/I_n \rightarrow \mathbb{Z}[q][x]/I_n^q$ that sends each coset $e_{i_1 \dots i_{n-1}}$ to the corresponding coset $E_{i_1 \dots i_{n-1}}$.

We can also directly define the quantum Schubert polynomials directly, though to some extent extrinsic, according to the result above, after knowing these standard elementary monomials indeed form a basis.

Definition 1.1. For an ordinary Schubert polynomial $\mathfrak{S}_w, w \in S_n$, if we write it as a linear combination: $\mathfrak{S}_w = \sum \alpha_{i_1 \dots i_{n-1}} e_{i_1 \dots i_{n-1}}$, then the corresponding quantum Schubert polynomial

$$\mathfrak{S}_w^q = \sum \alpha_{i_1 \dots i_{n-1}} E_{i_1 \dots i_{n-1}}.$$

Then we can rephrase the above version of the theorem to another one, still involving the geometric object, which we will finally prove:

Theorem 1.2. Then polynomial $\mathfrak{S}_w^q$ as defined above, is a coset representative of the image of the corresponding Schubert class σ_w under the canonical isomorphism $\mathrm{QH}^*(Fl_n, \mathbb{Z}) \cong \mathbb{Z}[q_1, \dots, q_{n-1}][x_1, \dots, x_n]/I_n^q$.

The survey paper will be devoted to revisiting the proof of the main theorem above in the original paper[1] and will not cover the further discussion on the Gromov-Witten invariants. In most of the cases where I give a full proof instead of a sketch, I've tried to explain the proofs in a more detailed way or present them in a more naturally logical order according to my understanding. Most of the preliminaries and notations in this survey paper are covered in class so will be omitted.

The outline of the survey paper goes as follows. In section 2, we shall define the standard elementary monomials and their quantum analogue, and then show that they indeed form a basis of the space $\mathbb{Z}[x]/I_n$ and $\mathbb{Z}[q][x]/I_n^q$, so that they can be used to define the quantization map to obtain the quantum Schubert polynomials. Then in section 3, we will introduce the family of commuting difference operators and we'll see that the correspondence between the quantum Schubert polynomials and the original ones could be algebraically carried out explicitly via these operators and we'd give the quantum Monk's rule, which is, in fact, just equivalent to the property of such operators. Next we'll list some of the necessary lemmas and use them to prove the important orthogonality by direct calculation in section 4, though will keep only the sketch of proofs for brevity. And we shall use the orthogonality to finally prove the main theorem in section 5.

2 Standard Monomials and Quantumization

Definition 2.1 (Elementary Symmetric Polynomial). The i 'th elementary symmetric polynomial is $e_i^k = e_i(x_1, \dots, x_k) =$

$$\sum_{1 \leq r_1 < \dots < r_i \leq k} x_{r_1} \dots x_{r_i}.$$

And as always, $e_0^k = 1$ and $e_i^k = 0$ if $i > k$ or $i < 0$.

Definition 2.2 (Standard Elementary Monomial). We will call a product of elementary polynomials $e_{i_1}^1 e_{i_2}^2 \dots e_{i_m}^m := e_{i_1 \dots i_m} (0 \leq i_k \leq k)$ a standard elementary polynomial. This is just

equivalent to saying it is a product of elementary polynomials e_i^k with upper indices pairwise different (since i_k is allowed to be 0).

To prove that these standard elementary monomials indeed form a basis, it is necessary to know how we could efficiently change (in fact, increase) the upper indices to make sure that no repetition will occur in upper indices in the expansion of some polynomial. The following key lemma tells us how to realize this straightening process.

Lemma 2.1. For $i, j, k \geq 1$, we have the following relation:

$$e_i^k e_j^k = e_i^{k+1} e_j^k + \sum_{l \geq 1} e_{i-l}^{k+1} e_{j+l}^k - \sum_{l \geq 1} e_{i-l}^k e_{j+l}^{k+1}. \quad (3)$$

Proposition 2.1 (Straightening). The standard elementary monomials form a linear basis of the ring $\mathbb{Z}[x_1, x_2, \dots]$ of polynomial functions in infinitely many variables.

The proof of the lemma is based on the obvious recurrence formula

$$e_i^k = e_i^{k-1} + x_k e_{i-1}^{k-1}. \quad (4)$$

And we shall need another lemma that will also be frequently used in the computation later, concerning about the action of the divided difference operator on the standar elementary polynomials.

Lemma 2.2. For $k \neq l$, $\partial_l e_i^k = 0$. Moreover, when $k \neq l$, ∂_l commutes with multiplication by e_i^k . When $k = l$, $\partial_k e_i^k = e_{i-1}^{k-1}$.

We shall omit the proof of the lemmas and the proposition here and do the similar work in the analogous but slightly more complicated quantum case below.

Definition 2.3. The quantum elementary polynomial E_i^k is the coefficient of λ^i in the polynomial $\det(I + \lambda G_k)$ where the matrix

$$G_k = \begin{pmatrix} x_1 & q_1 & 0 & \cdots & 0 \\ -1 & x_2 & q_2 & \cdots & 0 \\ 0 & -1 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_k \end{pmatrix}$$

Notice that this is equivalent to saying that E_i^k is the sum of i 'th principal minors of the matrix G_k . And as always, $E_i^k = 0$ if $i > k$ or $i < 0$.

To repeat the works we've done before, let's start from developing a similar recurrence formula for the quantum elementary polynomials. This can be done by the principal minor definition above if we classify the i 'th principal minors in G_k as those without x_k and those containing x_k , then expanding the latter along the last column will give us the formula (It's also possible to form a combinatorial model to explain the formula as in the original paper, which we'd omit here):

$$E_i^k = E_i^{k-1} + x_k E_{i-1}^{k-1} + q_{k-1} E_{i-2}^{k-2}. \quad (5)$$

Lemma 2.3. For $k \geq j \geq 0, k \geq i \geq 0$,

$$\begin{aligned} E_i^k E_j^k &= E_i^{k+1} E_j^k + \sum_{l \geq 1} E_{i-l}^{k+1} E_{j+l}^k - \sum_{l \geq 1} E_{i-l}^k E_{j+l}^{k+1} \\ &\quad - q_k \left(\sum_{l \geq 1} E_{i-1-l}^{k-1} E_{j-1+l}^k - E_{i-l}^k E_{j-2+l}^{k-1} \right). \end{aligned} \quad (6)$$

Proof. Let's first prove that for $k \geq j \geq 0, k \geq i \geq 0$,

$$E_i^k E_{j+1}^{k+1} + E_{i+1}^k E_j^k + q_k E_{i-1}^{k-1} E_j^k = E_j^k E_{i+1}^{k+1} + E_{j+1}^k E_i^k + q_k E_{j-1}^{k-1} E_i^k. \quad (7)$$

In fact this is direct from the recurrence formula. We have

$$\begin{aligned} E_i^k (E_{j+1}^{k+1} - E_{j+1}^k) &= E_i^k (x_{k+1} E_j^k + q_k E_{j-1}^{k-1}) \\ E_j^k (E_{i+1}^{k+1} - E_{i+1}^k) &= E_j^k (x_{k+1} E_i^k + q_k E_{i-1}^{k-1}). \end{aligned} \quad (8)$$

After subtracting the second equation from the first one, we would obtain the formula above.

Then rewrite the formula above we have

$$\begin{aligned} (E_{i+1}^k - E_{i+1}^{k+1}) E_j^k &= (E_{j+1}^k - E_{j+1}^{k+1}) E_i^k - q_k (E_{i-1}^{k-1} E_j^k - E_{j-1}^{k-1} E_i^k) \\ &= (E_i^k - E_i^{k+1}) E_{j+1}^k + E_i^{k+1} E_{j+1}^k - E_{j+1}^{k+1} E_i^k + (q - \text{terms}) \end{aligned}$$

Then induction of the first term on i will prove the original formula. $\square$

Lemma 2.2 has a more complicated form due to the asymmetric terms containing q_i .

Lemma 2.4. For $k \neq l, \partial_l E_i^k$ is a polynomial with degree $\leq i-3$. When $k \neq l, \partial_l$ no longer directly commutes with multiplication by E_i^k , but differ from a polynomial remainder. When $k = l, \partial_k E_i^k = e_{i-1}^{k-1} + \text{a polynomial with degree } \leq i-3$.

Proof. The first and the last assertement come directly from the observation that E_i^k has the highest term e_i^k and any other term whose coefficient contains q_i has lower degree (as a polynomial in x_i). Then ∂_i will do the same job on this highest term e_i^k as the Lemma 2.2, while when it will at least decrease the degree of the other terms by 1.

The second assertment follows from the Leibniz formula:

$$\partial_i(fE_i^k) = (\partial_i f) \cdot E_i^k + (s_i f) \partial_i E_i^k \quad (9)$$

while we've discussed about the last term above. $\square$

Theorem 2.1. The quantum standard elementary monomials form a linear basis in the ring $\mathbb{Z}[q][x_1, x_2, \dots]$ of quantum polynomial functions in infinitely many variables.

Proof. The first part is to prove that the quantum standard elementary monomials span the whole space. Write $x_i = E_1^i - E_1^{i-1}$, then we can write an arbitrary polynomial as some linear combination of the products of some E_i^k . Then we shall pick any of the monomials in the combination to work the following process to make sure there is no repetition in the upper indices. The process is done inductively on the degree of the polynomial and the repeated indice k (inversely). First, pick out the minimal repeated index k . Suppose the monomial contains $E_i^k E_j^k$. We can just replace $E_i^k E_j^k$ with the right-hand-side in the lemma above.

Then for the new terms without q_k , one of the upper indices increase to $k + 1$ while the other remains to be k . We can repeat this process if some of the obtained monomials still have k as a repeated index, until there is only one k in the upper indices of all the monomials and the repeated indices will then strictly increase, $\geq k + 1$, then we can just inductively carry out this process to eliminate all the repeated indices.

On the other hand, consider the new terms obtained containing q_k , the upper indices we've got is $k - 1$ and k , so we must be careful about that when carrying out the process again for them, we might still have k as a repeated indice after several steps. However, fortunately this situation won't last long, since every time we do the substitution as in Lemma 2.3, only the remainder polynomial terms with q_k get upper indices $k - 1$. Their degree (as polynomials in x_i), nevertheless, -2 every time we do the substitution. So after enough steps of substitutions, the remainder term will eventually become 0 and we just return to the situation as above. (Comparing with the Lemma 2.1 in the case of original standard elementary poly-

mials, the only difference is the remainder term, so the proof there is just the same with no worry as in this paragraph.)

Then we shall prove that these quantum Schubert polynomials are linearly independent. Suppose there exist a linear relation R with the minimal total degree. Suppose some E_i^k is a term appearing in this relation with minimal upper index k . Consider the action of ∂_k . This leads to degree strictly decreasing for those terms without any E_j^k by Lemma 2.4, and for those with E_i^k , $\partial_k(E_i^k f) = \partial_k f \cdot E_i^k + (s_k f) \cdot \partial_k E_i^k$, where f is a quantum standard elementary polynomial with indices strictly bigger than k , hence $s_k f = f$. So by the degree discussion in the Lemma 2.4 again we shall see that the action of ∂_k will give us a such relation with smaller total degree, a contradiction! Hence we have proved their linear independence. $\square$

It's direct by a dimensional counting argument to refine the proposition before into the finite variable case that we originally concern about:

Proposition 2.2. The quantum standard elementary monomials $E_{i_1 i_2 \dots i_{n-1}}$ as well as the monomials $x_1^{a_1} \dots x_{n-1}^{a_{n-1}}$ with $0 \leq a_k \leq n - k$, form a $\mathbb{Z}[q]$ -linear basis of $\mathbb{Z}[q, x] / I_n^q$.

Then we can define the quantum Schubert polynomial via the obvious correspondence between the standard elementary basis and that after quantization:

Definition 2.4. For an ordinary Schubert polynomial $\mathfrak{S}_w$, $w \in S_n$, if we write it as a linear combination: $\mathfrak{S}_w = \sum \alpha_{i_1 \dots i_{n-1}} e_{i_1 \dots i_{n-1}}$, then the corresponding quantum Schubert polynomial

$$\mathfrak{S}_w^q = \sum \alpha_{i_1 \dots i_{n-1}} E_{i_1 \dots i_{n-1}} \quad (10)$$

It is direct from this definition and the result in the ordinary case that the quantum Schubert polynomials $\mathfrak{S}_w^q$ also form a basis of the quantum cohomology ring $\mathbb{Z}[q, x] / I_n^q$. We shall omit the concrete example for brevity.

3 Commuting Difference Operators

Before digging into the most important orthogonal relation of the quantum Schubert polynomials, we will introduce a more practical viewpoint of the quantization map, using a tool of a family of commuting difference operators. This tool will be needed later to work

out the computations for the quantum Schubert polynomials. It is also straightforward with it to carry out the quantum Monk's rule through our computation.

First let's introduce some notations:

Definition 3.1. We shall use x_k in this chapter and later to both represent a polynomial and the operator that acts by the multiplication of x_k ;

We shall denote $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$. Note that $q_{ij} q_{jk} = q_{ik}$.

We shall also denote

$$\partial_{(ij)} = \partial_{t_{ij}} = \partial_i \partial_{i+1} \cdots \partial_{j-2} \partial_{j-1} \partial_{j-2} \cdots \partial_{i+1} \partial_i. \quad (11)$$

With these notations, let's introduce the difference operators $\mathcal{X}_k, k = 1, 2, \dots, n$:

Definition 3.2.

$$\mathcal{X}_k = x_k - \sum_{1 \leq i < k} q_{ik} \partial_{(ik)} + \sum_{k < j \leq n} q_{kj} \partial_{(kj)}. \quad (12)$$

More generally, for $\lambda_1, \dots, \lambda_n \in \mathbb{Z}[q]$,

$$\sum_i \lambda_i \mathcal{X}_i = \sum_i \lambda_i x_i + \sum_{1 \leq i < j \leq n} (\lambda_i - \lambda_j) q_{ij} \partial_{(ij)}. \quad (13)$$

We will need the following lemmas telling us about the commutativity between the terms in most general cases, of which we'd also omit the relatively easy proof, in which $[,]$ stands for the commutator.

Lemma 3.1. We have the following formulas:

1. For $a < c$, $[\partial_{(ac)}, x_b] = 0$ unless $a \leq b \leq c$.
2. For $a < b$, $[\partial_{(ab)}, x_a + x_{a+1} + \cdots + x_b] = 0$.
3. For $a < b, c < d$, $[\partial_{(ab)}, \partial_{(cd)}] = 0$ unless $b = c$ or $a = d$.
4. For $a < b < c$, $[\partial_{(ac)}, x_b] + [\partial_{(ab)}, \partial_{(bc)}] = 0$.

Then we could prove with the lemma the somehow striking fact that

Theorem 3.1. The operators $\mathcal{X}_1, \dots, \mathcal{X}_n$ commute pairwise.

Proof. Directly expand $[\mathcal{X}_a, \mathcal{X}_b]$ and omit the 0 terms according to the properties 1 and 3 in the lemma, we have

$$\begin{aligned}
[\mathcal{X}_a, \mathcal{X}_b] &= \left[x_a, - \sum_{i \leq a} q_{ib} \partial_{(ib)} \right] + \left[\sum_{j \geq b} q_{aj} \partial_{(aj)}, x_b \right] + \sum_{i < a} q_{ib} [\partial_{(ia)}, \partial_{(ab)}] \\
&\quad + \sum_{j > b} q_{aj} [\partial_{(ab)}, \partial_{(bj)}] - \sum_{a < i < b} q_{ab} [\partial_{(ai)}, \partial_{(ib)}] \\
&= -q_{ab} [x_a, \partial_{(ab)}] + q_{ab} [\partial_{(ab)}, x_b] + q_{ab} \sum_{a < i < b} [\partial_{(ab)}, x_i] \\
&= q_{ab} [\partial_{(ab)}, x_a + x_{a+1} + \dots + x_b] \\
&= 0.
\end{aligned}$$

We eliminate most of the terms at the second equality, using the property 4. □

Knowing the commutativity, the polynomial in these operators will be well-defined: for a polynomial $f \in \mathbb{Z}[q, x]$, we write $f(\mathcal{X})$ as the operator $f(\mathcal{X}_1, \dots, \mathcal{X}_n)$. A classical dimensional induction argument will give the following lemma:

Lemma 3.2. For any polynomial $f \in \mathbb{Z}[q, x]$, there exists a unique $F \in \mathbb{Z}[q, x]$ such that $f = F(\mathcal{X})(1)$.

We shall denote the correspondence $f \mapsto F$ as ψ , the crucial thing is that ψ is indeed the correct map that sends the Schubert polynomials to their quantum version.

Theorem 3.2. For any $w \in S_n$, $\psi(\mathfrak{S}_w) = \mathfrak{S}_w^q$, i.e., $\mathfrak{S}_w^q(\mathcal{X})(1) = \mathfrak{S}_w$.

In fact, for any standard elementary polynomial, we have $\psi(e_{i_1 \dots i_m}) = E_{i_1 \dots i_m}$.

The theorem is the straightforward corollary of the following proposition:

Proposition 3.1. We have $E_i^k(\mathcal{X})(g) = e_i^k g$ for any polynomial $g \in \mathbb{Z}[q, x]$ which is symmetric in the variables $x_1, \dots, x_{k+1}$, $k < n$, and when $g = 1$, $k = n$.

Proof. We can prove the theorem directly by induction on k . For $k = 0$, $E_0^0(\mathcal{X})(g) = 1 \cdot g = g$.

Then use the recurrence formula of the quantum Schubert polynomials, we have

$$\begin{aligned}
E_i^k(\mathcal{X})(g) &= E_i^{k-1}(\mathcal{X})(g) + \mathcal{X}_k E_{i-1}^{k-1}(\mathcal{X})(g) + q_{k-1} E_{i-2}^{k-2}(\mathcal{X})(g) \\
&= e_i^{k-1}(g) + \mathcal{X}_k(e_{i-1}^{k-1})(g) + q_{k-1} e_{i-2}^{k-2} g \\
&= e_i^{k-1}(g) + x_k e_{i-1}^{k-1} g - q_{k-1} \partial_{k-1} e_{i-1}^{k-1} g + q_{k-1} e_{i-2}^{k-2} g \\
&= e_i^{k-1} g + x_k e_{i-1}^{k-1} g \\
&= e_i^k g
\end{aligned}$$

□

Note that the operators $\mathcal{X}_i$ commute with the multiplication by the polynomials that are symmetric in $x_1, \dots, x_n$. This will lead to the following lemma.

Lemma 3.3. The map ψ is a bijection between I_n and I_n^q . The space $I_n \subset \mathbb{Z}[q, x]$ is an invariant subspace under the operators $\mathcal{X}_1, \dots, \mathcal{X}_n$.

Remark 3.1. In fact, in light of the observation of the invariant subspace, we might view $\mathcal{X}_i = \psi^{-1} x_i \psi$ as the operator which transports the multiplication by x_i in the space $\mathbb{Z}[q][x]/I_n^q$ to the dequantized space $\mathbb{Z}[q][x]/I_n$, so that we could perform $\mathcal{X}_i$ on the latter to understand the multiplication structure of the quantum cohomology ring.

It is proposition 3.1 that we really want the definition of the family of the commuting difference operators to satisfy, so that we could use them, at least for those polynomials with relatively good symmetry, to define the algebraic correspondence of the quantization process. The crucial step in the calculation above is to make sure $\mathcal{X}_k(e_{i-1}^{k-1} g) - x_k e_{i-1}^{k-1} g = -q_{k-1} \partial_{k-1} e_{i-1}^{k-1} g$ to ensure the recurrence process will be fine. This can be viewed as, to some extent, the naturality why we define such a family of operators. The mechanism is always to deal with the q -remainders. As discussed above, the action of $\mathcal{X}_i$ essentially define the multiplication in $\mathbb{Z}[q][x]/I_n^q$. The following quantum Monk's rule can be realized by this kind of thought, where the actual difference occurred just due to the q -remainders as in the commuting difference operators above.

Theorem 3.3 (Quantum Monk's Rule). For $w \in S_n$, $1 \leq r \leq n$, we have:

$$\mathfrak{S}_{s_r}^q \mathfrak{S}_w^q = (x_1 + \dots + x_r) \mathfrak{S}_w^q = \sum \mathfrak{S}_{t_{ab}}^q + \sum q_{cd} \mathfrak{S}_{t_{cd}}^q, \quad (14)$$

where the first sum is over the transpositons t_{ab} satisfying $a \leq r < b$ and $l(wt_{ab}) = l(w) - 1$ while the second is over the transpositons t_{cd} satisfying $c \leq r < d$ and $l(wt_{cd}) = l(w) - l(t_{cd})$.

More generally, for any f that is a $\mathbb{Z}[q]$ -linear combination of $x_i : f = \sum_i \lambda_i x_i$, we have:

$$f\mathfrak{S}_w^q = \sum (\lambda_a - \lambda_b)\mathfrak{S}_{t_{ab}}^q + \sum (\lambda_c - \lambda_d)q_{cd}\mathfrak{S}_{t_{cd}}^q, \quad (15)$$

where the first sum is over the transpositons t_{ab} satisfying $a < b$ and $l(wt_{ab}) = l(w) - 1$ while the second is over the the transpositons t_{cd} satisfying $c < d$ and $l(wt_{cd}) = l(w) - l(t_{cd})$.

Proof. From the remark above, the multiplication here can be considered as the action of $f(\mathcal{X})$ on $\mathfrak{S}_w$. Formally speaking, right action on both side of the formula above by $(\mathcal{X})(1)$ will turn it into a equivalent form concerning only about $\mathfrak{S}_w$.

On the other hand, by the action rule of the divided difference operators, we have

$$f(\mathcal{X})(\mathfrak{S}_w) = f \cdot \mathfrak{S}_w + \sum (\lambda_c - \lambda_d)q_{cd}\mathfrak{S}_{t_{cd}}^q, \quad (16)$$

where the sum is over the the transpositons t_{cd} satisfying $c < d$ and $l(wt_{cd}) = l(w) - l(t_{cd})$. Expansion of the first term according to the ordinary Monk's rule gives the equivalent form in the discussion above. $\square$

4 Orthogonality

In this section we shall sketch the proof of the orthogonal relation of the quantum Schubert polynomials and omit many of the computational details. The key method is to more carefully understand the action of the commuting difference operators $\mathcal{X}$ defined above on more general polynomials.

For $F \in \mathbb{Z}[q, x]/I_n^q$, let $\langle\langle F \rangle\rangle$ denote the coefficient of $\mathfrak{S}_{w_0}^q$ in the expansion of F in the basis of quantum Schubert polynomials. For our aim here, it's helpful to see that equivalently, the coefficient $\langle\langle F \rangle\rangle$ is the equal to the constant term of $\partial_{w_0}(F(\mathcal{X})(1))$.

Theorem 4.1. For $u, v \in S_n$,

$$\langle\langle \mathfrak{S}_u^q \mathfrak{S}_v^q \rangle\rangle = \begin{cases} 1 & \text{if } v = w_0 u \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

To sketch the proof, let's first notice that the only case to be proved that differs from the ordinary case is when $l(u) + l(v) > l(w_0)$. Then expand the Schubert polynomials as linear combinations of the quantum standard elementary monomials, we only need to prove that

$$\langle\langle E_{i_1 \dots i_{n-1}} E_{j_1 \dots j_{n-1}} \rangle\rangle = 0, \quad \forall i_1 + \dots + i_{n-1} + j_1 + \dots + j_{n-1} > l(w_0) = \frac{n(n-1)}{2} \quad (18)$$

Then use the equivalent definition of $\langle\langle, \rangle\rangle$ as above, we only need to show the constant term of the following is 0:

$$\partial_{w_0} (E_{i_1 \dots i_{n-1}}(\mathcal{X}) E_{j_1 \dots j_{n-1}}(\mathcal{X})(1)) = \partial_{w_0} (E_{i_1 \dots i_{n-1}}(\mathcal{X})(e_{j_1 \dots j_{n-1}})). \quad (19)$$

Brutally expand $E_{i_1 \dots i_{n-1}}(\mathcal{X})$ into the products of $(e_{i_m}^m + E_{i_m}^m(\mathcal{X}) - e_{i_m}^m)$, the only term with mere $e_{i_k}^k$ contributes 0 as in the ordinary case. While to deal with the other terms, we shall use the following stronger lemma:

Lemma 4.1. For any $k < n$ and any f ,

$$\partial_{w_0} e_{i_1 \dots i_{k-1}} (E_{i_m}^m(\mathcal{X}) - e_{i_m}^m) f = 0$$

In fact, except for an technical lemma needed, the lemma in this flavor can be viewed to be essentially asking about the general difference of the quantization map defined by the commuting difference operators as in the section above, between $E_{i_m}^m(\mathcal{X})$ and $e_{i_m}^m$. To describe explicitly, given $1 \leq k \leq n$, we decompose $\mathcal{X}_l$ at the k 'th position: $\mathcal{X}_l = \tilde{\mathcal{X}}_l + \tilde{\tilde{\mathcal{X}}}_l$ with

$$\begin{aligned} \tilde{\mathcal{X}}_l &= x_l - \sum_{1 \leq j < l} q_{jl} \partial_{(jl)} + \sum_{l < j \leq k} q_{lj} \partial_{(lj)}, \\ \tilde{\tilde{\mathcal{X}}}_l &= \sum_{j > k} q_{lj} \partial_{(lj)}. \end{aligned} \quad (20)$$

Then we have the following proposition which is essential to the proof of the lemma above:

Proposition 4.1. For any polynomial $f \in \mathbb{Z}[q, x]$,

$$E_i^k(\tilde{\mathcal{X}}_l) = e_i^k f.$$

In particular, if $q_k = 0$, $E_i^k(\mathcal{X})$ coincides with the multiplication of e_i^k .

The details of the proofs of the lemmas and propositions will be omitted.

5 Proof of the Main Theorem

We'll finally prove the theorem in this section. Without knowing the details of the geometric cohomology ring, we shall denote the elements in $\mathbb{Z}[q, x]/I_n^q$ that represent the Schubert classes by $\mathcal{Q}_w$ and list some of their properties as lemmas below without proof (while the first three of them is relatively clear by definition), which will be the foundation of the following proof. In the original paper, the authors also asked about the questions whether these properties can be taken as the axiom that characterize the quantum Schubert polynomials, but this won't be covered in this survey paper.

Lemma 5.1 (Homogeneity). If we let $\deg(x_i) = 1, \deg(q_j) = 2$, then every $\mathcal{Q}_w$ is homogeneous of degree $l(w)$.

Lemma 5.2 (Classical Limit). Specializing $q_1 = q_2 = \dots = q_{n-1} = 0$, we have $\mathcal{Q}_w = \mathfrak{S}_w$.

Lemma 5.3 (Nonnegativity of Structure Constants). The multiplication structure constants of the ring $\mathbb{Z}[q, x]/I_n^q$ with respect to the basis $\mathcal{Q}_w$, belongs to $\mathbb{Z}_{\geq 0}[q]$.

The next lemma is first proved by Ciocan-Fontaine in [2], which is an analogue of the ordinary case that states the ordinary Schubert polynomial of a cycle is the elementary symmetric polynomial, which can be seen directly from the pipe dreams, but in fact we will only use some weaker corollary of this lemma. Before that, let's introduce the notation of the semiring (i.e., it's closed under multiplication and addition) $\text{QH}_{\geq 0}^*$, to be the set of all linear combinations of $\mathcal{Q}_w$ with coefficients in $\mathbb{Z}_{\geq 0}[q]$.

Lemma 5.4. For a cycle $w = s_{k-i+1} \dots s_k \in S_n$, the polynomial $\mathcal{Q}_w$ is E_i^k , the quantum elementary polynomial. In particular, every $E_i^k \in \text{QH}_{\geq 0}^*$.

Then we are ready to prove the main theorem.

Proof. In light of the lemma 5.1, if we try to expand $\mathfrak{S}_w^q$ with respect to the basis $\mathcal{Q}_u$, the only non-zero coefficients are contributed by those $\mathcal{Q}_u$ with $l(u) \leq l(v)$. This is seen by comparing of degree using lemma 5.1. More explicitly, the coefficient of $\mathcal{Q}_u$ with $l(u) < l(v)$ must be a polynomial in q_i with positive degree, so when specializing q_i to 0, the only non-zero coefficient is that of $\mathcal{Q}_w$. Then using lemma 5.2 we shall see it is just 1. So the task is just to show the terms involving $\mathcal{Q}_u$ with $l(u) < l(v)$ actually do not exist.

From proposition 2.2 and the following conclusion that quantum Schubert polynomials form a basis of $\mathbb{Z}[q, x]/I_n^q$, in conjunction with a comparison of degree again, we will see that the Schubert polynomials $\mathfrak{S}_w$, with $l(w) = l$, are related to the standard elementary monomials $e_{i_1} \cdots e_{i_{n-1}}$ with $l_1 + \cdots + l_{n-1} = l$, by a nondegenerate linear transformation. This also means that every $\mathfrak{S}_w$ should be in the expansion of some $e_{i_1} \cdots e_{i_{n-1}}$. Then being a product of the elementary symmetric polynomials, which themselves are Schubert polynomials of cycles, the standard elementary monomials are also linear combination of Schubert polynomials with non-negative coefficient. Then together with the observation before, we actually have the coefficients α_w below are positive:

$$\sum_{i_1 + \cdots + i_{n-1} = l} e_{i_1} \cdots e_{i_{n-1}} = \sum_{l(w)=l} \alpha_w \mathfrak{S}_w. \quad (21)$$

Quantumize this equation, using the property in lemma 5.4, we will see that

$$\sum_{l(w)=l} \alpha_w \mathfrak{S}_w^q \in \text{QH}_{\geq 0}^*. \quad (22)$$

Moreover, together with the discussion above in the first paragraph, $\sum_{l(w)=l} \alpha_w \mathfrak{S}_w^q$ is just the sum of $\sum_{l(w)=l} \alpha_w \mathcal{Q}_w$ and the linear combination of some $\mathcal{Q}_u$ with $l(u) < l(w)$, so in fact, we have

$$\sum_{l(w)=l} \alpha_w (\mathfrak{S}_w^q - \mathcal{Q}_w) \in \text{QH}_{\geq 0}^*. \quad (23)$$

Then we are almost there, since in view of the inner product according to the orthogonality, we only need to prove that $\langle \mathfrak{S}_{w_0 v}^q, \mathcal{Q}_w \rangle = 0$ for $l(v) < l(w) = l$. Expanding $\mathfrak{S}_{w_0 v}^q$ again into the standard elementary monomials in the same degree, we only need to prove that given $E_{j_1 \cdots j_{n-1}}$ with $j_1 + \cdots + j_{n-1} > l(w_0) - l$, we have $\langle E_{j_1 \cdots j_{n-1}}, \mathcal{Q}_w \rangle = 0$. Then just take the inner product below, which by definition $\in \mathbb{Z}_{\geq 0}[q]$ since $E_{j_1 \cdots j_{n-1}} \in \mathbb{Z}_{\geq 0}[q]$ as well:

$$\langle E_{j_1 \cdots j_{n-1}}, \left(\sum_{l(w)=l} \alpha_w (\mathfrak{S}_w^q - \mathcal{Q}_w) \right) \rangle. \quad (24)$$

Using the orthogonality that $\langle E_{j_1 \cdots j_{n-1}}, \mathfrak{S}_w^q \rangle = 0$, we shall directly see that

$$- \sum_{l(w)=l} \alpha_w \langle E_{j_1 \cdots j_{n-1}}, \mathcal{Q}_w \rangle \in \mathbb{Z}_{\geq 0}[q]. \quad (25)$$

But since we've already seen that the coefficients α_w are strictly positive, and it's all the same to see that $\langle E_{j_1 \cdots j_{n-1}}, \mathcal{Q}_w \rangle \in \mathbb{Z}_{\geq 0}[q]$ as well, hence the only possibility is that all of them equal to 0. This finishes the proof. $\square$

References

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- [2] I. Ciocan-Fontanine, *Quantum cohomology of flag varieties*, Intern. Math. Research Notices (1995), no. 6, 263-277.