

Preliminary on Symplectic Varieties and Lagrangian Fibration

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1 Symplectic varieties

Def 1.1. A normal quasi-projective variety V is called *symplectic* if the smooth locus V^{sm} admits a holomorphic symplectic form σ which extends to a holomorphic form on any resolution of V .



As a caveat when reading literature, this definition is far from the so-called “*irreducible symplectic variety*” as in Matsushita’s earlier papers such as [Mat97] which requires V to be compact, simply connected, and $H^0(V, \Omega_V^2)$ spanned by a nowhere degenerate 2-form σ . Though seemingly only differing from irreducibility in the name, the latter is much stronger, and in fact Matsushita showed that every fibration $V \rightarrow B$ where V is irreducible symplectic is Lagrangian in [Mat97] using the Beauville-Fujiki form (which we would list as an example later).

Next we shall state the basic property of symplectic variety concerning the singularity. For the sake of completeness we include the definition of some types of singularities.

Def 1.2. Let V be a variety over a field of characteristic 0 and $f : X \rightarrow V$ a resolution of singularities. We say that $f : X \rightarrow V$ is a *rational resolution* if $f_* \mathcal{O}_X = \mathcal{O}_V$, and $R^i f_* \mathcal{O}_X = 0$ for $i > 0$. We say that V has *rational singularities* if every resolution $f : X \rightarrow V$ is rational.

V is said to have *Gorenstein singularities* if it is Cohen-Macaulay and its canonical divisor K_V is Cartier (locally free of rank 1).

Now suppose V is a normal variety such that K_V is $\mathbb{Q}$ -Cartier. Let $f : X \rightarrow V$ be a resolution of singularities. V has *canonical singularities* if for every exceptional divisor E_i , the discrepancy $a_i \geq 0$.

Prop 1.1. (1) For a symplectic variety V , $\omega_V \cong \mathcal{O}_V$, where ω_V denotes the dualizing sheaf.
(2) A symplectic variety has rational Gorenstein singularities.

Proof. (1) Suppose dimension $V = r$ (as algebraic variety) then σ^r (on V^{sm}) is a non-vanishing holomorphic form generating $\omega_{V^{sm}}$ hence $\omega_{V^{sm}}$ is trivial, namely, $\omega_{V^{sm}} \cong \mathcal{O}_{V^{sm}}$. Now V being normal, $\text{codim}(V \setminus V^{sm}) \geq 2$, we deduce $\mathcal{O}_V \cong \omega_V$ from the fact that ω_V is reflexive, and that for a reflexive sheaf $\mathcal{F}$ we have $\mathcal{F} \cong j_*(j^* \mathcal{F})$ for open

immersion $j : U \hookrightarrow X$ with $\text{codim}(X \setminus U) \geq 2$. (Refer to [Har 80] Proposition 1.6). The same also holds for $\mathcal{O}_V$ (which may also be proved more directly using algebraic Hartogs), hence $\omega_V \cong \mathcal{O}_V$.

(2) By (1), the dualizing sheaf is trivial, hence the canonical divisor is 0 and in particular Cartier. Again, consider σ' and a resolution $\rho : X \rightarrow V$, and in particular, $\rho^* \sigma'$ extends holomorphically. So the discrepancies a_i in the formula $K_X = \rho^* K_V + \sum a_i E_i$ satisfies $a_i \geq 0$ since $\rho^* \sigma'$ has no poles along the exceptional divisors E_i . Hence the singularities are canonical. so we deduce that the singularities are rational by [KM98] Theorem 5.22, thus V itself is Cohen-Macaulay by [KM98] Lemma 5.12, implying the singularities are Gorenstein. $\square$

E.g. 1.1. There are certainly normal varieties carrying a holomorphic symplectic form on the smooth part, but of non-rational singularity.

Consider the surface $X \subset \mathbb{C}^3$ defined by $x^3 + y^3 + z^3 = 0$. Over the smooth part (i.e., except the origin) we have the holomorphic 2-form given by Poincaré residue $\frac{1}{3x^2} dx \wedge dy = \frac{1}{3x^2} dy \wedge dz = \frac{1}{3y^2} dz \wedge dx$. X being a surface, this is automatically non-degenerate and closed.

But if we blow up the origin o , the exceptional divisor is the elliptic curve $E = \{x^3 + y^3 + z^3 = 0\} \subset \mathbb{P}^2$, hence for this blow-up $\rho : \tilde{X} \rightarrow X$, $(R^1 \rho_* \mathcal{O}_{\tilde{X}})_o \cong \Gamma(X, R^1 \rho_* \mathcal{O}_{\tilde{X}}) \cong H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong \mathbb{C} \neq 0$.

(For the calculation above, one geometric way is to consider $\pi : \tilde{\mathbb{C}}^3 \rightarrow \mathbb{C}^3$ that is the blow-up of the origin in the ambient space $\mathbb{C}^3$. The exceptional divisor E' of π is isomorphic to $\mathbb{P}^2$. The restriction of π to the strict transform $\tilde{X}$ gives our resolution $\rho : \tilde{X} \rightarrow X$. The total transform of X splits as $\pi^* X = \tilde{X} + 3E'$.

Now we look at the short exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{\mathbb{C}}^3}(-\tilde{X}) \rightarrow \mathcal{O}_{\tilde{\mathbb{C}}^3} \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow 0.$$

Taking the long exact sequence of higher direct images $R^i \pi_*$ yields:

$$\dots \rightarrow R^1 \pi_* \mathcal{O}_{\tilde{\mathbb{C}}^3} \rightarrow R^1 \pi_* \mathcal{O}_{\tilde{X}} \rightarrow R^2 \pi_* \mathcal{O}_{\tilde{\mathbb{C}}^3}(-\tilde{X}) \rightarrow R^2 \pi_* \mathcal{O}_{\tilde{\mathbb{C}}^3} \rightarrow \dots$$

Since $\mathbb{C}^3$ is smooth and π is the blow-up of a smooth point, $\tilde{\mathbb{C}}^3$ has rational singularities, implying $R^i \pi_* \mathcal{O}_{\tilde{\mathbb{C}}^3} = 0$ for all $i > 0$. The sequence therefore reduces to an isomorphism: $R^1 \pi_* \mathcal{O}_{\tilde{X}} \cong R^2 \pi_* \mathcal{O}_{\tilde{\mathbb{C}}^3}(-\tilde{X})$. Now substituting $\tilde{X} \sim \pi^* X - 3E'$ and applying the projection formula, we obtain:

$$\begin{aligned} R^2 \pi_* \mathcal{O}_{\tilde{\mathbb{C}}^3}(-\tilde{X}) &= R^2 \pi_* \left(\mathcal{O}_{\tilde{\mathbb{C}}^3}(3E') \otimes \mathcal{O}_{\tilde{\mathbb{C}}^3}(-\pi^* X) \right) \\ &\cong R^2 \pi_* \mathcal{O}_{\tilde{\mathbb{C}}^3}(3E') \otimes \mathcal{O}_{\mathbb{C}^3}(-X) \end{aligned}$$

Since X is a globally defined principal divisor on $\mathbb{C}^3$, the line bundle $\mathcal{O}_{\mathbb{C}^3}(-X)$ is trivial. We are reduced to computing $R^2 \pi_* \mathcal{O}_{\tilde{\mathbb{C}}^3}(3E')$, which is a skyscraper sheaf supported at o . This can be done by iteratively taking the long exact sequence associated to the following short exact sequence:

$$0 \rightarrow \mathcal{O}_{\tilde{\mathbb{C}}^3}((k-1)E') \rightarrow \mathcal{O}_{\tilde{\mathbb{C}}^3}(kE') \rightarrow \mathcal{O}_{E'}(kE') \rightarrow 0.$$

The normal bundle of the exceptional $\mathbb{P}^2$ in the blow-up of a point in a smooth 3-fold is $\mathcal{O}_{\mathbb{P}^2}(-1)$, so $\mathcal{O}_{E'}(kE') \cong \mathcal{O}_{\mathbb{P}^2}(-k)$. We can settle the calculation by noticing that $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-n)) = 0$ for $n = 1, 2$ and $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-n))$ always vanishes.

Now after the calculation as carried out above, the stalk of $R^2\pi_*\mathcal{O}_{\mathbb{P}^2}(-3)$ is just $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}$.

Rmk 1.1. As a highly nontrivial result, the requirement for a symplectic variety that the 2-form σ extends holomorphically on any resolution can be equivalently altered to that X has rational Gorenstein singularities, which may seem more intuitive, see [Na 01].

2 Lagrangian fibration

Def 2.1 (Lagrangian subvariety). Let X be a symplectic variety with symplectic form σ . An irreducible subvariety $Y \subset X$, not contained in the singular locus of X , is called *isotropic* if $\omega|_{Y \cap X^{sm}}$ vanishes. Note that $\dim Y \leq \frac{1}{2} \dim X$ by elementary linear algebra. Y is called *Lagrangian* if the equality holds.

Def 2.2 (Lagrangian fibration). Let X be a symplectic variety with symplectic form σ , and B be a normal variety. A proper surjective morphism $f : X \rightarrow B$ with connected fibers is called a *Lagrangian fibration* if the generic fiber F of f is a Lagrangian subvariety of X with respect to σ , i.e. $\dim F = \frac{1}{2} \dim X$ and $\sigma|_{F \cap X^{sm}} = 0$.

E.g. 2.1 (Examples of Lagrangian Fibrations). (1) Hitchin fibration (note that the total space is not compact).
(2) A fibration from irreducible holomorphic symplectic manifold:

Def 2.3. An *irreducible holomorphic symplectic manifold* is a simply connected compact Kähler manifold X , such that $H^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic 2-form σ . (such as K3 surfaces, i.e., compact surface with trivial canonical bundle and $b^{1,0} = 0$, and Kummer surface, and two examples of O'Grady).

Thm 2.1 (Matsushita 1997). Let X be an irreducible holomorphic symplectic manifold of dimension $2n$, and $f : X \rightarrow B$ a proper surjective morphism to a normal variety B such that $0 < \dim B < \dim X$, and a general fiber of f is connected, then f is a Lagrangian fibration. Here I only state the result as an example without proof. Time permitting, we would prove this in later talks.

(3) a Lagrangian fibration with all singular fibers irreducible but non-reduced:

We construct a quasi-projective symplectic surface admitting a proper surjective Lagrangian fibration over $\mathbb{A}_{\mathbb{C}}^1$, such that the singular fiber consists of a single irreducible component which is everywhere non-reduced.

Let the ambient space be $\mathcal{Y} = \mathbb{P}_{[x_0:x_1]}^1 \times \mathbb{P}_{[y_0:y_1]}^1 \times \mathbb{A}_v^1$, and let $Q = x_0y_1 - x_1y_0$. The vanishing locus $Q = 0$ cuts out the diagonal $\mathcal{I} \cong \mathbb{P}^1$ inside $\mathbb{P}^1 \times \mathbb{P}^1$. Let $H(x, y)$ be a generic bi-homogeneous polynomial of bidegree $(2, 2)$ in the projective coordinates. (In fact, one can check that $H(x, y) = x_0^2y_0^2 + x_1^2y_1^2 + 2x_0^2y_1^2 + 2x_1^2y_0^2$ is an appropriate choice that is “generic” to satisfy all the transversality and smoothness conditions later in the construction.)

Define the total space $X \subset \mathcal{Y}$ as the hypersurface given by the equation: $F := Q^2 - v \cdot H(x, y) = 0$. Being hypersurface defined by homogeneous polynomial, it is singular at points where F and all partial derivatives of F simultaneously vanish. One can calculate that this requires $H = 0, Q = 0$, and for generic choice of H such that $\nabla F \neq 0$ (such as the choice as shown above), $v = 0$ as well, and there are exact 4 singular points for generic H .

We first verify that X admits a symplectic form on its smooth locus. The ambient space $\mathcal{Y}$ is smooth, with canonical bundle: $K_{\mathcal{Y}} \cong \mathcal{O}(-2, -2) \boxtimes \mathcal{O}_{\mathbb{A}^1}$ (where $\mathcal{O}(-2, -2) = \mathcal{O}(-2) \boxtimes \mathcal{O}(-2)$ is the canonical bundle on $\mathbb{P}^1 \times \mathbb{P}^1$). The hypersurface X is the zero locus of a global section of $\mathcal{O}(2, 2) \boxtimes \mathcal{O}_{\mathbb{A}^1}$. By the adjunction formula, the canonical bundle of X is:

$$K_X \cong (K_{\mathcal{Y}} \otimes \mathcal{O}_{\mathcal{Y}}(X))|_X \cong (\mathcal{O}(-2, -2) \otimes \mathcal{O}(2, 2))|_X \cong \mathcal{O}_X.$$

Because $\dim X = 2$ and its canonical bundle is trivial, X possesses a global, nowhere-vanishing holomorphic 2-form σ . As long as H is chosen generically (so that X has only isolated singularities, such as the explicit choice given above), σ restricts to a valid symplectic form on the smooth locus X^{sm} .

Now, being a hypersurface in a smooth variety, X is necessarily Cohen-Macaulay, also we have checked explicitly that X is regular in codimension 1, hence it is normal. Also, passing over to complete local ring (or equivalently, viewing in complex analytic category and consider the corresponding local neighborhood), one may see that the singularity of X is of A_1 -type, which is rational. Hence X is indeed a symplectic variety.

Now define the projection morphism $f: X \rightarrow \mathbb{A}^1_v$ by mapping points to their v -coordinate. Since X is a closed subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{A}^1$, the projection onto the $\mathbb{A}^1$ factor is a proper morphism.

The generic fibers when $v \neq 0$ are curves. In fact, they are smooth curves of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, by Bertini Theorem as long as H is generic. (By the genus formula for complete intersections in $\mathbb{P}^1 \times \mathbb{P}^1$: $g = (d_1 - 1)(d_2 - 1) = (2 - 1)(2 - 1) = 1$ Thus, the generic fiber is a smooth proper elliptic curve, agreeing with Thm 2.4 below.) The restriction of the 2-form σ to any 1-dimensional fiber is trivially zero for dimension reasons. Thus, the generic fiber is a Lagrangian subvariety.

Now when $v = 0$, the defining equation becomes:

$$Q^2 = (x_0 y_1 - x_1 y_0)^2 = 0$$

This scheme is supported entirely on the diagonal $\mathcal{A} \cong \mathbb{P}^1$. There are no other irreducible components. Due to the exponent, the scheme-theoretic fiber is $2\mathcal{A}$. The fiber has exactly one irreducible component, and it carries a multiplicity of 2 everywhere.

Now we turn to the first important property of Lagrangian fibration, proved in [Ma00]. The essential input of this theorem is the following general result by Kollár.

Thm 2.2 (Kollár). Let $f: X \rightarrow B$ be a proper surjective morphism from a smooth Kähler manifold to a normal variety. Then $R^i f_* \omega_X$ is torsion free.

Thm 2.3 (Matsushita). Let $f : X \rightarrow B$ be a Lagrangian fibration from a symplectic variety to a normal variety B , then every irreducible component of every fiber of f is a Lagrangian subvariety.

Proof. We first explain that how to reduce to the smooth case, which is settled originally in Matsushita's paper. This actually depends on the fact that the singularities are rational. Pick a resolution $\rho : Y \rightarrow X$: since X and Y are Cohen-Macaulay, by Grothendieck duality we find that $\rho_* \omega_Y = \omega_X$ and $R^i \rho_* \omega_Y = 0$ for $i > 0$. (In the identity $R\rho_* R\mathcal{H}om(\mathcal{F}^\bullet, \omega_Y^\bullet) = R\mathcal{H}om(R\rho_* \mathcal{F}^\bullet, \omega_X^\bullet)$, take $\mathcal{F}^\bullet = \mathcal{O}_Y$ and use that since X, Y are both Cohen-Macaulay, $\omega_Y^\bullet = \omega_Y[\dim Y]$, $\omega_X^\bullet = \omega_X[\dim X]$). Now suppose we have a fibration $f : X \rightarrow B$, by Leray spectral sequence we may find that $R^i f_*(\rho_* \omega_Y) = R^i f_* \omega_X$. (By $E_2^{p,q} = R^p f_*(R^q \rho_* \omega_Y) \Rightarrow R^{p+q}(f \circ \rho)_* \omega_Y$, since $R^q \rho_* \omega_Y = 0$ for $q > 0$, $E_2^{p,q} = 0$ for all $q > 0$, and $E_2^{p,0} = R^p f_* \omega_X$, and thus differentials in E_2 page are all trivial. Hence $E_2^{p,q} = E_\infty^{p,q}$ and in particular $R^i(f \circ \rho)_* \omega_Y = \bigoplus_{p+q=i} E_\infty^{p,q} = E_2^{i,0} = R^i f_* \omega_X$).

In a word, $R^i(f \circ \rho)_* \omega_Y = R^i f_* \omega_X$, so in particular is torsion free by Kollár's Theorem. Also recall that in our case $\omega_X \cong \mathcal{O}_X$, so $R^i f_* \mathcal{O}_X$ is torsion free.

Consider the conjugate $\bar{\sigma}$ of the symplectic form $\sigma \in H^0(X, \Omega_X^2)$, and $\bar{\sigma}$ can be viewed as an element in $H^2(X, \mathcal{O}_X) = H^2(X, \Omega_X^0) = \overline{H^0(X, \Omega_X^2)}$ via Hodge symmetry. By the Leray spectral sequence $H^p(B, R^q f_* \mathcal{O}_X) \Rightarrow H^{p+q}(X, \mathcal{O}_X)$, we get a morphism $u : H^2(X, \mathcal{O}_X) \rightarrow H^0(B, R^2 f_* \mathcal{O}_X)$ by composing $H^2(X, \mathcal{O}_X) \rightarrow E_\infty^{0,2} \hookrightarrow E_2^{0,2} = H^0(B, R^2 f_* \mathcal{O}_X)$. Now notice that $R^2 f_* \mathcal{O}_X$ being torsion free exactly implies that $u(\bar{\sigma}) = 0$. Indeed, $u(\bar{\sigma})$ is a section of $R^2 f_* \mathcal{O}_X$ that vanishes generically: (we may pick the generic point b where f is flat and the fiber is Lagrangian, then by Grauert's Theorem, $u(\bar{\sigma})(b) = u|_{X_b}(\bar{\sigma}|_{X_b}) = 0$, and by Nakayama's lemma we see $u(\bar{\sigma}) \equiv 0$), hence a torsion section and thus 0. Now consider any fiber X_t of f , and a resolution $\tilde{X}_t$ of X_t .

Then consider the composition of morphisms $H^2(X, \mathcal{O}_X) \rightarrow H^0(B, R^2 f_* \mathcal{O}_X) \rightarrow (R^2 f_* \mathcal{O}_X)_t \rightarrow H^2(X_t, \mathcal{O}_{X_t}) \rightarrow H^2(\tilde{X}_t, \mathcal{O}_{\tilde{X}_t})$, where the second map is given by restricting the section to the fiber over t and the third is the base change morphism, then the image of $\bar{\sigma}$ under the whole morphism is 0 since $u(\bar{\sigma})$ is already 0. Thus, the image of $\bar{\sigma}$ in $H^2(\tilde{X}_t, \mathcal{O}_{\tilde{X}_t}) = \overline{H^0(\tilde{X}_t, \Omega_{\tilde{X}_t}^2)}$ is 0, and in particular $\sigma|_{X_t} = 0$.

Thus X_t is isotropic, so $\dim X_t \leq \frac{1}{2} \dim X = \dim B$. But due to upper semi-continuity of fiber dimension, we have $\dim X_t \geq \dim X - \dim B = \dim B$, so it is Lagrangian. $\square$

Cor 2.1. Let $f : X \rightarrow B$ be a Lagrangian fibration where B is smooth, then f is flat.

The corollary directly follows from the following proposition, known as so-called ‘‘Miracle Flatness’’, since we have seen that a symplectic variety X is Cohen-Macaulay in Prop 1.1. (For a proof, see [Nag62].)

Prop 2.1. Let $A \rightarrow B$ be a local morphism of noetherian local rings. Assume that A is regular, B is Cohen-Macaulay, and $\dim B = \dim A + \dim B/\mathfrak{m}_A B$, then $A \rightarrow B$ is flat.

Next we come to the second property of Lagrangian fibration, whose proof of this property actually comes directly from the lemma.

Thm 2.4. Any smooth fiber of a Lagrangian fibration $f : X \rightarrow B$ is an abelian variety.

Lem 2.1. Any compact Kähler manifold Y with trivial tangent bundle is a complex torus.

Proof. We use Albanese morphism $\alpha_Y : Y \rightarrow \text{Alb}(Y) = \frac{H^0(Y, \Omega_Y^1)^*}{H_1(Y, \mathbb{Z})}$, defined by $\alpha_Y(y) = (\theta \mapsto \int_{y_0}^y \theta) \pmod{H_1(Y, \mathbb{Z})}$ for a fixed basepoint y_0 and $\theta \in H^0(Y, \Omega_Y^1)$.

Then $d\alpha_Y : T_y Y \rightarrow T_{\alpha_Y(y)} \text{Alb}(Y) = H^0(Y, \Omega_Y^1)^*$ is simply the evaluating map $v \mapsto (\theta \mapsto \theta(v))$. Now since $\Omega_Y^1 \cong T_Y^*$ is trivial, $H^0(Y, \Omega_Y^1)$ has a basis $\theta_1, \dots, \theta_n$ with these θ_i everywhere linearly-independent, where $n = \text{rank } T_Y$.

So $d\alpha_Y$ is an isomorphism everywhere, hence α_Y is a local biholomorphism. Now since Y is compact and $\text{Alb}(Y)$ is Hausdorff (as complex analytic spaces), α_Y is automatically proper, and a proper local homeomorphism is a finite covering. And being a finite covering of a complex torus, Y itself is a complex torus as well. (After passing to fundamental group in view of topological Galois correspondence, this is just saying a finite index subgroup of $\mathbb{Z}^{2n}$ is still isomorphic to $\mathbb{Z}^{2n}$ and we are done.) $\square$

Proof of the Theorem. Being a fiber of a proper map, any fiber X_b over a point b is proper and hence X_b is compact (viewed as a complex analytic space). In view of the lemma above we only need to prove that T_{X_b} is trivial (since being a subvariety of a projective variety X , X_b will be an abelian variety as long as it is a complex torus).

Now suppose f is smooth over b . Without loss of singularity we may restrict ourselves to the smooth part of X . We first notice that this implies the conormal bundle $\mathcal{N}_{X_b/X}^*$ of X_b in X is trivial. (This is seen by pullback the exact sequence $0 \rightarrow f^* \Omega_B \rightarrow \Omega_X \rightarrow \Omega_{X/B} \rightarrow 0$ to the fiber X_b (i.e., tensoring with $\mathcal{O}_{X_b}$, which is left exact here since $\Omega_{X/B}$ is locally free) to get

$$0 \rightarrow \mathcal{O}_{X_b} \otimes T_{B,b}^* \cong \mathcal{O}_{X_b}^{\oplus n} \rightarrow \Omega_X|_{X_b} \rightarrow \Omega_{X_b} \rightarrow 0,$$

and compare with the conormal bundle exact sequence of the subvariety X_b inside X :

$$0 \rightarrow \mathcal{N}_{X_b/X}^* \rightarrow \Omega_X|_{X_b} \rightarrow \Omega_{X_b} \rightarrow 0.)$$

Now we settle the problem with the isomorphism $T_{X_b} \cong \mathcal{N}_{X_b/X}^*$ induced by the symplectic form since it vanishes identically on X_b . $\square$

3 Relative Dimension Bound

As is seen in previous lectures, when the total space is not smooth and we are pushing forward any complex K such as IC-sheaves instead of constant sheaves, to generalize support theorem as in [MS22], we need to require $\tau_{>2c}(Rf_* K) = 0$, where c is the relative dimension of a weak abelian fibration. We have a following geometric criterion for a equidimensional fibration to determine this vanishing.

Prop 3.1 (Relative Dimension Bound). Let $f : X \rightarrow B$ be an equidimensional proper morphism of complex algebraic varieties of relative dimension c . Suppose that there exists a Whitney stratification $X = \bigsqcup X_i$ such that for any $b \in B$ we have

$$\text{codim}_X X_i \geq 2 \text{codim}_{f^{-1}(b)}(X_i \cap f^{-1}(b)). \quad (1)$$

Then we have

- $\tau_{>2c}(Rf_* \text{IC}(X, \mathbb{Q})) = 0$;
- $R^{2c}f_* \text{IC}(X, \mathbb{Q}) = R^{2c}f_* \mathbb{Q}_X$.

Proof. Via proper base change theorem, taking the stalks of the higher direct image sheaves commutes with taking the cohomology of the fibers. Hence, we have

$$\mathcal{H}^*(Rf_* \text{IC}(X, \mathbb{Q}))_b = H^*(f^{-1}(b), \text{IC}(X, \mathbb{Q})|_{f^{-1}(b)}) \quad \text{for any } b \in B.$$

These cohomology groups are the limit of the Grothendieck (hypercohomology) spectral sequence:

$$E_2^{p,q} = H^{p-q}(f^{-1}(b), \mathcal{H}^q(\text{IC}(X, \mathbb{Q}))|_{f^{-1}(b)}) \Rightarrow H^p(f^{-1}(b), \text{IC}(X, \mathbb{Q})|_{f^{-1}(b)}).$$

Let $Z_q \subseteq X$ be the support of the constructible sheaf $\mathcal{H}^q(\text{IC}(X, \mathbb{Q}))$. Because the complex dimension of the restriction is $\dim_{\mathbb{C}}(Z_q \cap f^{-1}(b))$, its real topological dimension is $2 \dim_{\mathbb{C}}(Z_q \cap f^{-1}(b))$. Cohomology on this space vanishes in degrees higher than its real dimension. In particular, we have:

$$E_2^{p,q} = 0 \quad \text{for } p - q > 2 \dim(Z_q \cap f^{-1}(b)).$$

The irreducible components of Z_q are the closure of some X_i with $q < \text{codim}_X X_i$, or X itself, if $q = 0$. Indeed, the strong support condition for intersection cohomology implies that $\mathcal{H}^q(\text{IC}(X, \mathbb{Q}))_x = 0$ for any $x \in X_i$ and $q \geq \max\{\text{codim}_X X_i, 1\}$.

We can rewrite condition to bound the dimensions. Since f is equidimensional of relative dimension c , $\text{codim}_{f^{-1}(b)}(X_i \cap f^{-1}(b)) = c - \dim(X_i \cap f^{-1}(b))$. Substituting this into (??) yields $2 \dim(X_i \cap f^{-1}(b)) \leq 2c - \text{codim}_X X_i$.

By summing q to this inequality, we can bound the non-vanishing terms of the spectral sequence:

$$q + 2 \dim(Z_q \cap f^{-1}(b)) \leq q + 2c - \text{codim}_X X_i.$$

For $q > 0$, the support condition forces $q \leq \text{codim}_X X_i - 1$. Therefore:

$$q + 2 \dim(Z_q \cap f^{-1}(b)) \leq (\text{codim}_X X_i - 1) + 2c - \text{codim}_X X_i = 2c - 1.$$

Thus, we conclude that $E_2^{p,q} = 0$ for $p > q + 2 \dim(Z_q \cap f^{-1}(b))$, which means $E_2^{p,q}$ vanishes for $p > 2c$ and arbitrary q , or for $p = 2c$ and $q \neq 0$. This degeneration immediately yields:

$$\mathcal{H}^{>2c}(Rf_* \text{IC}(X, \mathbb{Q})) = 0, \quad \mathcal{H}^{2c}(Rf_* \text{IC}(X, \mathbb{Q})) = \mathcal{H}^{2c}(Rf_* \mathbb{Q}_X).$$

□

Prop 3.2. Let X be a symplectic variety endowed with a Lagrangian fibration $f : X \rightarrow B$. Then X admits a complex Whitney stratification

$$X = \bigsqcup X_i$$

by locally closed disjoint algebraic symplectic submanifolds X_i such that the restriction map $f|_{X_i} : X_i \rightarrow B_i := f(X_i)$ is Lagrangian. In particular, for any $b \in B$ we have

$$\operatorname{codim}_X X_i = 2 \operatorname{codim}_{f^{-1}(b)} (X_i \cap f^{-1}(b)).$$

Proof. By [Kal06, Thm 2.3, Prop. 3.1], X admits a stratification by locally closed subsets X_i , with $i = 1, \dots, r$, such that:

- X_i are smooth and symplectic;
- X_i is an open subset of the singular locus of the closure of some X_j ;
- the stratification is analytically trivial in the normal direction to each stratum, so it is a Whitney stratification by Lemma 2.2.

Applying [Mat15, Thm 3.1] iteratively, the restrictions $f|_{X_i}$ are Lagrangian.

Since both f and $f|_{X_i}$ are Lagrangian, we obtain $\dim X - \dim X_i = 2 \dim f^{-1}(b) - 2 \dim (X_i \cap f^{-1}(b))$. □