

Notes on Special Kähler Geometry

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0 Introduction

The main task here is to introduce the special Kähler structure on the moduli space $\mathcal{M}$. The discussion will be mainly separated into two parts: the first being some explicit calculation in our case of Calabi-Yau 3-folds while the second being some discussion about general (projective) special Kähler condition.

Before we introduce the definition of special Kähler geometry, I would discuss about some notational issues first since I would follow the paper [S] of Strominger for the explicit calculation in the case of Calabi-Yau 3-fold. However there are some difference between his general setting and the setting in our paper.

In a word, what Strominger considers in his paper is a moduli space $\mathcal{M}$ of complex structures over a given Calabi-Yau manifold with of Hodge bundle $\mathcal{H}$ whose fibers $\mathcal{H}_x$ are $H^3(Y_x, \mathbb{C})$, $x \in \mathcal{M}$ where Y_x is the Calabi-Yau 3-fold corresponding to the point $x \in \mathcal{M}$. While in our paper we can consider more concretely as $\mathcal{M}$ being a kind of global toric variety and Y_x being a family of hypersurfaces.

We also consider a $(3,0)$ -form $\Omega_x = R_x(\omega_0)$ where R_x is the residue map that allows us to transmit from a form Ω_x on Y_x to a form ω_0 on the ambient toric variety which is easier to calculate. Correspondingly, Strominger uses such a “global” form, i.e., a section of the bundle $\mathcal{H}$, Ω which he simply takes to be the form on $\mathcal{M}$ locally given by Ω_x .

Over the moduli $\mathcal{M}$, this is only well-defined up to projective transformations which are not constant on $\mathcal{M} : \Omega \xrightarrow{f} e^{f(x)} \Omega$. We shall explain that speaking rigourously, picking Ω

gives a section of a bundle $\mathcal{H} \otimes L$ instead of a section of $\mathcal{H}$ itself, where L is an appropriate line bundle.

Recall the Weil-Petersson metric defined by $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K}$, where $\mathcal{K}(x, \bar{x}) = -\log \langle \Omega_x, \overline{\Omega_x} \rangle$. Note that the Kähler potential $\mathcal{K}$ is not defined globally but only locally, and further, over adjacent patches U_1, U_2 , suppose $\Omega \rightarrow e^{f_{21}(x)} \Omega$, then the Kähler potential $\mathcal{K}$ transforms as $\mathcal{K}_2 = \mathcal{K}_1 - f_{21} - \overline{f_{21}}$. Thus there exists a real line bundle R such that the transition function of a section s of R between two adjacent patches s_1, s_2 is given by $s_1 = e^{-(f_{21} + \overline{f_{21}})} s_2$. We shall show that this line bundle can be globally decomposed as $L^{-1} \otimes \overline{L}^{-1}$ for certain L .

This essentially relies on a result of Tian (see Theorem 2. in [T], p.641) that the Kähler form associated to this metric $\omega = -\frac{\sqrt{-1}}{2\pi} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is integral, i.e., represents a class in $H^2(\mathcal{M}, \mathbb{Z})$. Hence there exists a holomorphic hermitian line bundle (L, h) with hermitian metric $h = e^{\mathcal{K}}$. (This is a result combining Lefschetz Theorem on $(1,1)$ -classes and a proposition constructing suitable hermitian metric on the desired line bundle. Readers may refer to [GH] p.163, p.148, respectively.) In fact, Tian proved that this line bundle is exactly the tautological line bundle of the projective space $\mathbb{P}(H^3(Y_x, \mathbb{C}))$, as defined in our paper till the ending of section (2-1) and denoted by $\mathcal{L}$ there.

Now by the tautological construction of this line bundle L , it is essentially determined by its transition functions on the section Ω , namely, $\Omega \mapsto e^{f_{21}} \Omega$ on the intersection of patches U_1, U_2 . Now we can directly check that $R \cong L^{-1} \otimes \overline{L}^{-1}$ via calculation on transition functions. This is simply because

$$-\log \langle e^f \Omega, e^{\overline{f}} \overline{\Omega} \rangle = -\log \mathcal{K} - f - \overline{f}.$$

Recall last time we have constructed a “corrected” covariant derivative $D_i = \partial_i + K_i$ acting on Ω , where $K_i = \partial_i \mathcal{K} = -\frac{\langle \partial_i \Omega, \overline{\Omega} \rangle}{\langle \Omega, \overline{\Omega} \rangle}$, to ensure that it is compatible with the Hodge decomposition rather than merely the filtration. In fact since ∂_i may only mix a $(3,0)$ -form with a $(2,1)$ -form, we can calculate this “correction” term K_i reversely by the orthogonal relations between forms that are not of conjugate type. Thus we get a set of basis $D_i \Omega$ of $H^{2,1}(Y_x, \mathbb{C})$. We shall denote this set of basis by $G_i = D_i \Omega$, partially following the notation in [S]. From the discussion above we see that G_i is really a section of the bundle $T^* \otimes \mathcal{H} \otimes L$, where T^* is the holomorphic cotangent bundle.

We shall first prove an identity that the inner product of $G_i, G_{\bar{j}}$ gives minus the Kähler

metric on $T \otimes L^{-1}$, which is the first line in section (2-5) in our paper [Ho].

Prop 0.1. $\langle G_i, G_{\bar{j}} \rangle = -e^{-\mathcal{H}} g_{i\bar{j}}$.

Proof.

$$\begin{aligned}
-e^{-\mathcal{H}} g_{i\bar{j}} &= (-e^{-\mathcal{H}}) \left(-\partial_i \left(\frac{\langle \Omega, \partial_{\bar{j}} \bar{\Omega} \rangle}{\langle \Omega, \bar{\Omega} \rangle} \right) \right) \\
&= e^{-\mathcal{H}} \frac{\langle \Omega, \bar{\Omega} \rangle \langle \partial_i \Omega, \partial_{\bar{j}} \bar{\Omega} \rangle - \langle \partial_i \Omega, \bar{\Omega} \rangle \langle \Omega, \partial_{\bar{j}} \bar{\Omega} \rangle}{\langle \Omega, \bar{\Omega} \rangle^2} \\
&= \frac{\langle \Omega, \bar{\Omega} \rangle \langle \partial_i \Omega, \partial_{\bar{j}} \bar{\Omega} \rangle - \langle \partial_i \Omega, \bar{\Omega} \rangle \langle \Omega, \partial_{\bar{j}} \bar{\Omega} \rangle}{\langle \Omega, \bar{\Omega} \rangle} \\
&= \langle \partial_i \Omega, \partial_{\bar{j}} \bar{\Omega} \rangle - \frac{\langle \partial_i \Omega, \bar{\Omega} \rangle \langle \Omega, \partial_{\bar{j}} \bar{\Omega} \rangle}{\langle \Omega, \bar{\Omega} \rangle};
\end{aligned}$$

$$\begin{aligned}
\langle G_i, G_{\bar{j}} \rangle &= \langle \partial_i \Omega + K_i \cdot \Omega, \partial_{\bar{j}} \bar{\Omega} + K_{\bar{j}} \cdot \bar{\Omega} \rangle \\
&= \langle \partial_i \Omega, \partial_{\bar{j}} \bar{\Omega} \rangle - \frac{\langle \partial_i \Omega, \bar{\Omega} \rangle}{\langle \Omega, \bar{\Omega} \rangle} \langle \Omega, \partial_{\bar{j}} \bar{\Omega} \rangle - \frac{\langle \Omega, \partial_{\bar{j}} \bar{\Omega} \rangle}{\langle \Omega, \bar{\Omega} \rangle} \langle \partial_i \Omega, \bar{\Omega} \rangle + \frac{\langle \partial_i \Omega, \bar{\Omega} \rangle \langle \Omega, \partial_{\bar{j}} \bar{\Omega} \rangle}{\langle \Omega, \bar{\Omega} \rangle} \\
&= \langle \partial_i \Omega, \partial_{\bar{j}} \bar{\Omega} \rangle - \frac{\langle \partial_i \Omega, \bar{\Omega} \rangle \langle \Omega, \partial_{\bar{j}} \bar{\Omega} \rangle}{\langle \Omega, \bar{\Omega} \rangle}.
\end{aligned}$$

□

Recall and summarize what we have till now. The inner product of the 3-forms $(\Omega, G_i, G_{\bar{i}}, \bar{\Omega})$ with the symplectic basis α_I, β^J defines the period matrix, hence defining $2r + 2$ elements forming a basis of the bundle $E = L \oplus (T^* \otimes L) \oplus (\bar{T}^* \otimes \bar{L}) \oplus \bar{L}$, where T^* is the cotangent bundle of $\mathcal{M}$ while L is a holomorphic hermitian line bundle over $\mathcal{M}$ with curvature $-2\pi i \omega$, namely, whose Chern class is exactly $[\omega]$. A Kähler manifold equipped with such a line bundle will be called a *Hodge manifold* in later sections. To avoid confusion with our chosen $(3,0)$ -form Ω , we denote the $(2r + 2) \times (2r + 2)$ period matrix as

$$\begin{pmatrix} \varpi \\ \bar{\varpi} \end{pmatrix} = \begin{pmatrix} D_i X^I & D_i P^J \\ \overline{D_i X^I} & \overline{D_i P^J} \end{pmatrix}.$$

1 Covariant Derivatives and Related Calculations

In this section we will discuss about The covariant derivatives appeared in our paper [Ho]. Readers may find that there are differently defined covariant derivatives all denoted by D_i in this paper. I shall clarify this and carry out the calculation of some basic properties of them, to be prepared for the proof (which is essentially calculation again) for the main theorems.

Def 1.1. We define $D_i^{(k,l)}$ to be the covariant derivative in the holomorphic direction ∂_i associated to the metric connection acting on the bundle $T^* \otimes L^{\otimes k} \otimes \bar{L}^{\otimes l}$.

In the terminology of our paper [Ho], this means that $D_i^{(k,l)}$ is the metric-compatible covariant derivative for a section of weight (k, l) . This could be similarly extended to bundles such as $\text{Sym}^a T^* \otimes L^{\otimes k} \otimes \bar{L}^{\otimes l}$ for later use. However, these upper indices (k, l) are not explicit displayed in the text of our paper. For example, in section (3-1), when acting on the prepotential $\mathcal{F}(X)$, which is a section of $\text{Sym}^3 T^* \otimes L^{\otimes 2}$, the formula is shown as in formula (3.4), while in the case before, for Ω which is a section of $L = \text{Sym}^0 T^* \otimes L$, the formula is $D_i^{(1,0)} = \partial_i + K_i$. So to clarify this, we shall first do the following calculation. Similar calculation could also be done for general $\text{Sym}^a T^* \otimes L^{\otimes k} \otimes \bar{L}^{\otimes l}$. For a discussion in the physical setting, the reader may refer to the paper [B], p.34 (in arXiv version).

Prop 1.1.

$$D_i^{(m,0)} = \partial_i + mK_i - \Gamma_{i*}^*.$$

$$\text{More explicitly, } D_i^{(m,0)} V^k = \partial_i V_k^k + mK_i \cdot V_k^k - \Gamma_{ik}^j V_j^k.$$

Proof. The Kähler metric over $T \otimes L^{-1}$ is given by $h_{i\bar{j}} = -\langle G_i, G_j \rangle = e^{-\mathcal{K}} g_{i\bar{j}}$, and similarly over $T \otimes L^{-m}$ is given by $-e^{-m\mathcal{K}} g_{i\bar{j}}$. Hence the corresponding Kähler metric connection is given by

$$\begin{aligned} (e^{m\mathcal{K}} g^{k\bar{l}}) \partial_i (e^{-m\mathcal{K}} g_{j\bar{i}}) &= e^{m\mathcal{K}} g^{k\bar{l}} (-e^{-m\mathcal{K}} \partial_i \mathcal{K} g_{j\bar{i}} + e^{-m\mathcal{K}} \partial_i g_{j\bar{i}}) \\ &= -m \partial_i \mathcal{K} g^{k\bar{l}} g_{j\bar{i}} + \Gamma_{ik}^j = -mK_i \delta_j^k + \Gamma_{ik}^j. \end{aligned}$$

Taking the induced connection on the dual bundle of $T \otimes L^{-1}$, i.e., $T^* \otimes L$, implies the result. $\square$

We can well do similar calculations to prove the correct covariant derivatives D_i acting on $\overline{T}^* \otimes \overline{L}$ is ∂_i , which means the expression of D_i before Theorem 2.1 in our paper [Ho] is reasonable, as well as $\overline{D}_{\bar{i}}$.

Next let's deduce some basic commutativity properties of these covariant derivatives, to be used in calculations later.

Prop 1.2.

$$\begin{aligned} [D_i, D_j] &= 0; \\ [D_i, \overline{D}_{\bar{j}}]\Omega &= -g_{i\bar{j}}\Omega; \\ [D_i, \overline{D}_{\bar{j}}]\overline{\Omega} &= g_{i\bar{j}}\overline{\Omega}. \end{aligned}$$

Proof. First as a remark, the first property actually holds for any metric connections D_i defined as above, such as those acting on $\text{Sym}^3 T^* \otimes L^{\otimes 2}$ and this will be of basic use in later sections, but we shall only calculate the case when they act on sections of weight $(1, 0)$ here, since the calculation in the general case is analogous. Also we will not exhibit the calculation of the third formula, being the same as that of the second one.

Then by definition, $D_i G_k = \partial_i G_k + \partial_i \mathcal{K} \cdot G_k - \Gamma_{ij}^k G_j$, hence

$$\begin{aligned} D_l D_i G_k &= \partial_l (\partial_i G_k) + \partial_l \mathcal{K} \cdot \partial_i G_k - \Gamma_{lj}^k \partial_i G_j \\ &\quad + \partial_i (\partial_l \mathcal{K} \cdot G_k) + \partial_l \mathcal{K} \cdot \partial_i \mathcal{K} \cdot G_k - \partial_i \mathcal{K} \cdot \Gamma_{lj}^k G_j \\ &\quad - \Gamma_{ij}^k \partial_l G_j - \partial_l \mathcal{K} \cdot \Gamma_{ij}^k G_j + \Gamma_{ij}^k \Gamma_{ls}^j G_s. \end{aligned}$$

Similarly one obtain a formula for $D_i D_l G_k$ and one may notice that all the terms involving Γ_{i*}^* are symmetric in i and l , while the sum of the other four terms in $D_i D_l G_k$ and $D_l D_i G_k$ are also the same, both being equal to

$$\partial_l \partial_i G_k + \partial_l \mathcal{K} \cdot \partial_i \mathcal{K} \cdot G_k + \partial_l \partial_i \mathcal{K} \cdot G_k + \partial_l \mathcal{K} \cdot \partial_i G_k + \partial_i \mathcal{K} \cdot \partial_l G_k,$$

finishing the proof of the first property.

Next consider $[D_i, \overline{D}_{\bar{j}}]\Omega$. First notice that Ω is holomorphic, hence $\partial_{\bar{j}}\Omega = 0$ and thus $\overline{D}_{\bar{j}}\Omega = 0$. Also $\partial_{\bar{j}}\partial_i\Omega = \partial_i\partial_{\bar{j}}\Omega = 0$. Hence

$$[D_i, \overline{D}_{\bar{j}}]\Omega = -(\partial_{\bar{j}}\partial_i\Omega + \partial_{\bar{j}}(\mathcal{K}_{\bar{i}} \cdot \Omega)) = -\partial_{\bar{j}}(\partial_i \mathcal{K}) \cdot \Omega = -g_{i\bar{j}}\Omega.$$

□

Rmk 1.1. You can ignore the boring calculation above as soon as we have proposition 1.1. and you realize that the curvature matrix of the metric connection on a hermitian bundle is an hermitian matrix of $(1,1)$ -forms, which is a general result in complex geometry that you can refer to in [GH] p.75. This means that if we separate the curvature 2-form into three parts according to type: $\Omega = \Omega^{(2,0)} + \Omega^{(1,1)} + \Omega^{(0,2)}$, then only the mixed type curvature does not vanish, which is given by the commutator of mixed covariant derivatives $[D_i, \overline{D}_{\bar{j}}]$, while two holomorphic or two anti-holomorphic covariant derivatives always commute.

One may naturally ask that how these D_i 's act on this bundle, such as on the basis elements in $\mathcal{H}$ and $\overline{\mathcal{H}}$ especially. By the orthogonal conditions and the two calculational propositions 0.1 and 1.2, one can deduce this easily. In fact one can compute the connection matrix A_i that makes $\mathbb{D}_i = D_i + A_i$ acting covariantly on this set of basis, and since the transition function of the bundle is a constant matrix in Sp_{2r+2} , we know that we get a flat connection $\mathbb{D}$ on the bundle E (which acts as $\mathbb{D}_i$ on the basis element G_i).

Recall the Yukawa coupling $C_{ijk} = \sqrt{-1} \langle \Omega, \partial_i \partial_j \partial_k \Omega \rangle$. Denote $\mathcal{E}_{im}^{\bar{n}} = \sqrt{-1} e^{\mathcal{K}} C_{imn} g^{n\bar{m}}$ and $\overline{\mathcal{E}}_{\bar{i}\bar{m}}^n = -\sqrt{-1} e^{\mathcal{K}} \overline{C}_{\bar{i}\bar{m}\bar{n}} g^{n\bar{n}}$. Then we have the following:

Thm 1.1. The period matrix satisfies:

$$(D_i + A_i) \begin{pmatrix} \mathcal{H} \\ \overline{\mathcal{H}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad ; \quad (\overline{D}_{\bar{i}} + \overline{A}_{\bar{i}}) \begin{pmatrix} \mathcal{H} \\ \overline{\mathcal{H}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} (A_i)_{0,n} &= -\delta_i^n, \quad (A_i)_{m,\bar{n}} = \mathcal{E}_{im}^{\bar{n}}, \quad (A_i)_{\bar{m},0} = -g_{i\bar{m}} \\ (\overline{A}_{\bar{i}})_{0,\bar{n}} &= -\delta_{\bar{i}}^{\bar{n}}, \quad (\overline{A}_{\bar{i}})_{\bar{m},n} = \overline{\mathcal{E}}_{\bar{i}\bar{m}}^n, \quad (\overline{A}_{\bar{i}})_{m,0} = -g_{i\bar{m}}, \end{aligned}$$

and the other terms vanish.

Proof. We shall only calculate $(A_i)_{m,\bar{n}}$ since the other calculations are similar, using the other properties obtained in the proposition above. Then suppose (remembering $G_i := D_i \Omega$ is our notation)

$$D_i(D_j \Omega) = c_0 \Omega + c_m D_m \Omega + d_m \overline{D}_{\bar{m}} \overline{\Omega} + d_0 \overline{\Omega}.$$

Then since the only non-vanishing inner product is $\langle D_n \Omega, \overline{D}_{\bar{m}} \overline{\Omega} \rangle = -e^{-\mathcal{K}} g_{n\bar{m}}$ as proven in proposition 0.1. We find that $c_0 = d_0 = c_m = 0$ and

$$-d_m e^{-\mathcal{K}} g_{k\bar{m}} = \langle D_k \Omega, D_i D_j \Omega \rangle.$$

Considering the type of the forms we may see that

$$\int_{\mathcal{M}} \Omega \cup \partial_i \partial_j \Omega = 0.$$

Differentiating this equation by ∂_k , we immediately find that

$$\int_{\mathcal{M}} \partial_k \Omega \cup \partial_i \partial_j \Omega = - \int_{\mathcal{M}} \Omega \cup \partial_i \partial_j \partial_k \Omega = C_{ijk}.$$

The similar result holds for the covariant derivatives D_i, D_j, D_k . Hence

$$-d_m e^{-\mathcal{K}} g_{k\bar{m}} = -\langle \Omega, D_k D_i D_j \Omega \rangle.$$

However, again by arguing with the type of the forms, we can see that the extra terms $K_i + I_i^*$ make no contribution to the inner product, since one derivative of a $(3,0)$ -form at most increase the antiholomorphic rank by one and hence the only nonvanishing term in the inner product is contributed by $\langle \Omega, \partial_i \partial_j \partial_k \Omega \rangle$ and thus with the commutativity of D_k, D_i, D_j , we get $-d_m e^{-\mathcal{K}} g_{k\bar{m}} = \sqrt{-1} C_{ijk}$, namely,

$$(A_i)_{m,\bar{n}} = \mathcal{C}_{im}^{\bar{n}} = \sqrt{-1} e^{\mathcal{K}} C_{imn} g^{n\bar{n}}.$$

□

The connection $\mathbb{D} = D + A$ obtained as above is flat by the discussion ahead of the calculation. As a corollary, the curvature tensors vanish. This gives us nontrivial implications about the original covariant derivative D_i .

Thm 1.2. The vanishing of $[\mathbb{D}_i, \mathbb{D}_j]$ is equivalent to the existence of a (not necessarily holomorphic) prepotential; while the vanishing of the mixed type $[\mathbb{D}_i, \overline{\mathbb{D}}_j]$ is equivalent to the curvature condition that

$$-R_{i\bar{j}l}^k = g_{i\bar{j}} \delta_l^k + g_{j\bar{l}} \delta_i^k - e^{2\mathcal{K}} C_{ilm} \overline{C_{j\bar{k}\bar{m}}}^{m\bar{n}} g^{k\bar{k}}.$$

Proof. I shall only give a sketch of this proof, omitting the detail of the calculations.

For the first statement, consider the inner product

$$\begin{aligned} \langle G_k, [D_i, D_j] G_l \rangle &= \langle D_k \Omega, D_i D_j D_l \Omega \rangle - \langle D_k \Omega, D_j D_i D_l \Omega \rangle \\ &= \langle D_i \Omega, D_k D_j D_l \Omega \rangle - \langle D_i \Omega, D_k D_i D_l \Omega \rangle = D_j C_{ikl} - D_i C_{jkl}. \end{aligned}$$

Hence if this formula is equal to 0, we can use the Poincaré lemma iteratively to find that locally $C_{ijk} = D_i D_j D_k \mathcal{F}$ (but this $\mathcal{F}$ is not necessarily holomorphic indeed).

For the second statement, expand the following equation properly:

$$0 = \langle G_l, [\mathbb{D}_i, \overline{\mathbb{D}_j}] G_k \rangle.$$

□

We shall end this section of computations by introducing the notations in section (2-6) in our paper [Ho]. First denote $\alpha = (0, i)$ and $\bar{\alpha} = (\bar{0}, \bar{i}), i = 0, \dots, r$ for the indices of the basis of $\mathfrak{g}$ and their conjugates. Recall that we can normalize the period matrix, i.e., $\mathfrak{g} = (D_\alpha X^I \ D_\alpha P^J) = (D_\alpha \ X^I)(I_{r+1} \ \tau_{IJ})$, where I_{r+1} is the $(r+1) \times (r+1)$ identity matrix and $\tau_{IJ} = \frac{\partial^2}{\partial X^I \partial X^J} \mathcal{F}$. Also denote $\xi_\alpha^I := D_\alpha X^I$ and similarly for $\xi_I^\alpha, \bar{\xi}_{\bar{\alpha}}^I, \bar{\xi}_I^{\bar{\alpha}}$. (You may check [Ho] for the concrete definition if confused.)

Then one may naturally how the computational results proven above change as the base change. First, notice the result in proposition 0.1 can be summarized as following

(where $Q = \begin{pmatrix} 0 & I_{r+1} \\ -I_{r+1} & 0 \end{pmatrix}$ as in the Riemann bilinear relation):

$$\begin{pmatrix} g_{0\bar{0}} & 0 \\ 0 & g_{i\bar{j}} \end{pmatrix} = -e^{\mathcal{K}} \sqrt{-1} \mathfrak{g} Q^t \bar{\mathfrak{g}} \quad (g_{0\bar{0}} \equiv -1).$$

Then

$$g_{\alpha\bar{\beta}} = -\sqrt{-1} e^{\mathcal{K}} \xi_\alpha^I (I_{r+1} \ \tau_{IJ}) Q^t \overline{(I_{r+1} \ \tau_{IJ})} \bar{\xi}_{\bar{\beta}}^J$$

and notice that

$$(I_{r+1} \ \tau_{IJ}) Q^t \overline{(I_{r+1} \ \tau_{IJ})} = (-\tau_{IJ} \ I_{r+1}) \begin{pmatrix} I_{r+1} \\ \tau_{IJ} \end{pmatrix} = (\bar{\tau} - \tau)_{IJ}.$$

Hence we get that

$$g_{\alpha\bar{\beta}} = \sqrt{-1} e^{\mathcal{K}} \xi_\alpha^I (\tau - \bar{\tau})_{IJ} \bar{\xi}_{\bar{\beta}}^J.$$

Therefore it might be natural to introduce a metric by $\mathcal{G}_{IJ} = \sqrt{-1} e^{\mathcal{K}} (\tau - \bar{\tau})_{IJ}$ and similarly its inverse $\mathcal{G}^{IJ}$ so that we have $g_{\alpha\bar{\beta}} = \xi_\alpha^I \mathcal{G}_{IJ} \bar{\xi}_{\bar{\beta}}^J$. For the results in theorem 2.1 reformulated in the notations ξ_i , readers may refer to (2.13) in our paper [Ho] directly since I do not want to copy here.

2 (Projective) Special Kähler Manifold

First let us recall our bundle $E = L \oplus (T^* \otimes L) \oplus (\overline{T^*} \otimes \overline{L}) \oplus \overline{L}$ and its basic properties mentioned in the intro section. This will firstly lead to the following definition. (The definitions and results in this section are all from Freed's paper [F] and readers may find many details therein.)

Def 2.1. A *Hodge manifold* is a triple (M, L, ω) where M is a Kähler Manifold and $L \rightarrow M$ is a holomorphic hermitian line bundle over M with curvature $-2\pi i\omega$, namely, whose Chern class is exactly $[\omega]$.

Moreover, as in our case, we shall directly give the definition of projective special Kähler manifold.

Def 2.2. A *projective special Kähler structure* on an n -dim Hodge manifold (M, L, ω) is a triple (V, ∇, Q) where

- (i) $V \rightarrow M$ is a holomorphic vector bundle of rank $n + 1$ with a given holomorphic inclusion $L \hookrightarrow V$;
- (ii) ∇ is a flat connection on the underlying real bundle $V_{\mathbb{R}}$ such that $\nabla(L) \subseteq V$ and the section $m \mapsto L_m$ which is a map from M to $\mathbb{P}((V_{\mathbb{R}})_{\mathbb{C}})$ is an immersion with respect to ∇ ;
- (iii) Q is a nondegenerate skew form on $V_{\mathbb{R}}$ which has type $(1, 1)$ with respect to the complex structure and satisfies $\nabla Q = 0$. Furthermore, we assume that $Q|_{L \times \overline{L}}$ is $\frac{i}{2\pi}$ times the Hermitian metric on L .

In our case, V is just $L \oplus (T^* \otimes L)$ as described in section 0, and the flat connection $\mathbb{D}$ satisfying a certain Griffiths' transversality is also shown to exist in section 1. As for the nondegenerate skew form Q , it is just the natural pairing $\langle \cdot, \cdot \rangle$.

Hence we can easily find that this definition is well-suited for our case. However, we shall introduce another seemingly shorter definition of the special Kähler manifold in non-projective case and prove that there is certain correspondence between these two cases.

Def 2.3. A *special (pseudo-)Kähler manifold* is a complex manifold (M, J) where J is a certain complex structure with

(i) a (pseudo-)Kähler metric g with (pseudo-)Kähler form ω , where pseudo means the metric may not be positive definite;

(ii) a flat symplectic torsion-free connection ∇ which is “compatible” with the complex structure J , that is to say (which we shall explain later),

$$\nabla^2 = 0, \quad \nabla \omega = 0, \quad d_{\nabla}(\text{id}) = 0, \quad d_{\nabla}J = 0.$$

Rmk 2.1. Let me clarify the conditions in this definition. (If you are not familiar with these things as me, you may find further reference in [Hu], section 4.2, and definitions in papers [A], [Bi], [H], besides Freed’s paper [F] itself. In fact, this remark is just an integration of these materials.)

Before we begin, first note that in general for a vector bundle E over a real manifold, the connection ∇ gives a natural extension of the de Rham complex:

$$0 \rightarrow \mathcal{A}^0(E) \xrightarrow{d_{\nabla} := \nabla} \mathcal{A}^1(E) \xrightarrow{d_{\nabla}} \mathcal{A}^2(E) \rightarrow \dots$$

Here $\mathcal{A}^k(E)$ denotes the sheaf of E -valued k -forms. And an extension $\nabla : \mathcal{A}^k(E) \rightarrow \mathcal{A}^{k+1}(E)$ is defined as follows: for a local k -form α and a local section of E , we define $\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla(s)$.

In our case we simply take the vector bundle E to be the tangent bundle TM . Besides, we view id and the complex structure J , being an endomorphism of the tangent bundle, as 1-forms with values in T . So when we write $d_{\nabla}J, d_{\nabla}(\text{id})$, they genuinely live in $\mathcal{A}^2(TM)$, while $\nabla \omega \in \mathcal{A}^3(TM)$.

(i) After this discussion, we can write out these conditions in a more explicit way via “covariant derivatives”. For example, $d_{\nabla}J$ as a 2-form, when acting on two tangent vectors X, Y , we have

$$d_{\nabla}J(X, Y) = (\nabla_X J)Y - (\nabla_Y J)X.$$

Hence we see that $d_{\nabla}J = 0$ is not really saying $\nabla_X J \equiv 0$ as 1-form and is certainly weaker than this, rather, it feels like certain symmetry. In fact, since the Kähler form ω and the complex structure J determines the Kähler metric g , one can see if we require $\nabla_X \equiv 0$ in a strong sense, we can only get the Levi-Cevita connection, which is too restrictive.

Similarly, the torsion-free condition reads

$$d_{\nabla}(\text{id})(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

(ii) If one denotes $\pi^{(1,0)} \in \mathcal{A}^{1,0}(T_{\mathbb{C}}M)$ to be the projection onto the $(1,0)$ part of the complexified tangent bundle, i.e., $\pi^{(1,0)} = \frac{\text{id} - iI}{2}$, then the condition $d_{\nabla}J = 0$ is equivalent to $d_{\nabla}\pi^{(1,0)} = 0$ on the complexified tangent bundle. One may see from this description that the condition of compatibility with the complex structure implies the condition of torsion-freeness, inasmuch as id is twice the real part of $\pi^{(1,0)}$.

(iii) Finally let's discuss about the symplectic condition. In fact, for any symplectic manifold, we can require a connection to be torsion-free and compatible with the symplectic structure ω . Similar to the discussion before, the latter means that

$$(\nabla_X \omega)(Y, Z) := X(\omega(Y, Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z) = 0.$$

One may wonder about the existence and uniqueness of such a symplectic connection, and how it would add stronger constraint by requiring compatibility with an extra complex structure and flatness. We would summarize the existence results of symplectic connections in the proposition below, stating there exists non-unique symplectic connections, parametrized by certain symmetric 3-tensor, while on the other hand, flatness is a much stronger requirement such that one can determine the whole geometry of the manifold under this special Kähler assumption via a single holomorphic function called prepotential.

Prop 2.1. There always exist symplectic connections on a symplectic manifold (M, ω) . The set of symplectic connections is an affine space modelled on the space of contravariant symmetric 3-tensor fields $I^{\infty}(\text{Sym}^3 TM)$ on M .

Proof. To prove the existence, we first take any torsion-free connection ∇^0 such as the Levi-Civita connection. Consider a tensor N on M defined by $\nabla_X^0 \omega(Y, Z) =: \omega(N(X, Y), Z)$, being well-defined by the nondegeneracy of ω . And since ω is skew symmetric we have $\omega(N(X, Y), Z) = -\omega(N(X, Z), Y)$ and since ω is closed, we know the cyclic sum $\sum_{\text{cyc}} \omega(N(X, Y), Z) = 0$. (This is from the identity that can be checked directly with the definition of $d\omega$ and the torsion-free condition for ∇_X^0 that $d\omega(X, Y, Z) = (\nabla_X^0 \omega)(Y, Z) + (\nabla_Y^0 \omega)(Z, X) + (\nabla_Z^0 \omega)(X, Y)$.) Thus we define

$$\nabla_X Y := \nabla_X^0 Y + \frac{1}{3}N(X, Y) + \frac{1}{3}N(Y, X).$$

We can directly observe that this is torsion-free and

$$\begin{aligned}
\nabla_X \omega(Y, Z) &= X(\omega(Y, Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z) \\
&= \nabla_X^0 \omega(Y, Z) - \frac{1}{3} \omega(N(X, Y), Z) - \frac{1}{3} \omega(N(Y, X), Z) \\
&\quad - \frac{1}{3} \omega(Y, N(X, Z)) - \frac{1}{3} \omega(Y, N(Z, X)) \\
&= \nabla_X^0 \omega(Y, Z) - \frac{1}{3} \omega(N(X, Y), Z) - \frac{1}{3} \omega(N(Y, X), Z) \\
&\quad - \frac{1}{3} \omega(N(X, Y), Z) - \frac{1}{3} \omega(N(Z, Y), X) \\
&= \frac{1}{3} (\omega(N(X, Y), Z) + \omega(N(X, Z), Y)) = 0,
\end{aligned}$$

where the third equation comes from $\omega(Y, N(X, Z)) = -\omega(N(X, Z), Y) = \omega(N(X, Y), Z)$ and the last one comes from the cyclic sum relation we deduced above.

Thus we see this ∇_X is symplectic. And suppose ∇'_X is also symplectic with $\nabla'_X Y = \nabla_X Y + S(X, Y)$. Then the torsion-free condition is equivalent to that $S(X, Y) = S(Y, X)$ and by

$$\nabla'_X \omega(Y, Z) = \nabla_X \omega(Y, Z) - \omega(S(X, Y), Z) - \omega(Y, S(X, Z)) = \omega(S(X, Z), Y) - \omega(S(X, Y), Z),$$

we find that $\nabla'_X \omega = 0$ if and only if $\omega(S(X, Y), Z)$ is totally symmetric. $\square$

After these clarifications, we shall come to the main result in this section, namely the “correspondence” between the projective and non-projective cases. For this, we consider a principal $\mathbb{C}^\times : \widetilde{M} \rightarrow M$ on a Hodge manifold M obtained by deleting the zero section of $L \rightarrow M$. Then the Hermitian connection on L is also a connection on $p : \widetilde{M} \rightarrow M$, i.e., a $\mathbb{C}^\times$ -invariant distribution of horizontal subspaces $\ker dp$. Also, we have a tautological nonzero holomorphic section s of $p^*L \rightarrow \widetilde{M}$ by setting $s(x) = x$ which is a section of the fiber $L_{p(x)}$. Then we need first know how to lift the Kähler metric.

Lem 2.1. Let $\tilde{\omega} \in \Omega^{1,1}(\widetilde{M})$ satisfy that $\tilde{\omega}(Y_x, Z_x) = |s_x|^2 p^* \omega(Y_x, Z_x)$ for two horizontal vectors $Y_x, Z_x \in \ker dp$, vanishes when pairing a horizontal vector with a vertical one, and is $\frac{-1}{\pi}$ times the canonical Kähler form on pairs of vertical vectors. Then we have $\tilde{\omega} = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} |s|^2$ and thus $\tilde{\omega}$ is closed in particular. Hence we get a pseudo-Kähler metric on $\widetilde{M}$. Finally, $p^* \omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |s|^2$.

In particular we see from this construction that the metric $\tilde{\omega}$ is negative on fibers and positive on horizontal subspaces, hence has signature $(n, 1)$ where $n = \dim M$. We shall omit the proof and readers may refer to [F], Lemma 4.1 directly.

Thm 2.1. Let (M, L, ω) be a Hodge manifold with associated pseudo-Kähler manifold $(\tilde{M}, \tilde{\omega})$ and the tautological section s .

(i) A projective special Kähler structure on M induces a $\mathbb{C}^\times$ -invariant special pseudo-Kähler structure $\tilde{\nabla}$ on $(\tilde{M}, \tilde{\omega})$ with $\tilde{\nabla}s = \pi^{(1,0)}$ is the projection onto the $(1, 0)$ -part of the complexified tangent bundle.

(ii) Conversely, a $\mathbb{C}^\times$ -invariant special pseudo-Kähler structure $\tilde{\nabla}$ on $(\tilde{M}, \tilde{\omega})$ satisfying $\tilde{\nabla}s = \pi^{(1,0)}$ induces a projective special pseudo-Kähler structure on (M, L, ω) .

Proof. (i) Let $\tilde{\nabla} = p^*\nabla$. Using the inclusion $L \hookrightarrow V$ we may view s as a section of p^*V . The Griffiths' transversality condition for the projective special Kähler condition implies that $\tilde{\nabla}s \in p^*V$ and the immersion condition implies that $\tilde{\nabla} : T\tilde{M} \rightarrow p^*V$ is an isomorphism.

Considering the underlying real structure under this isomorphism, we get a real flat connection on $\tilde{M}$ denoted also by $\tilde{\nabla}$. Also, under $\tilde{\nabla}s$, $p^*Q := \tilde{Q}$ pulls back to $-\tilde{\omega}$ since if we differentiate the third condition of projective special Kähler structure that

$$\frac{i}{2\pi}|s|^2 = \tilde{Q}(s, \bar{s}),$$

we may obtain that

$$\tilde{\omega} = \frac{i}{2\pi}\bar{\partial}\partial|s|^2 = -\tilde{Q}(\tilde{\nabla}s, \tilde{\nabla}\bar{s}).$$

Further we see $\tilde{\nabla}\tilde{\omega} = 0$ from this. And under $\tilde{\nabla}s$, s corresponds to a holomorphic vector field ζ such that $\tilde{\nabla}\zeta = \pi^{(1,0)}$, hence $d_{\tilde{\nabla}}\pi^{(1,0)} = 0$, which is equivalent to the special Kähler condition as pointed out in remark 2.1.(ii).

(ii) The construction is just a reverse process of that above. Let now V be the quotient of $T\tilde{M}$ by the $\mathbb{C}^\times$ -action. Then it is a holomorphic bundle over M and the vertical vectors in $T\tilde{M}$ gives a line bundle L which includes into V . The connection $\tilde{\nabla}$ on $(T\tilde{M})_{\mathbb{R}}$ induces a connection ∇ on $V_{\mathbb{R}}$ under the quotient of the $\mathbb{C}^\times$ -action while the immersion condition comes from the hypothesis $\tilde{\nabla}s = \pi^{(1,0)}$. The form $-\tilde{\omega}$ on $\tilde{M}$ induces a skew form Q on V . $\square$

As soon as we find such a correspondence, one may expect that the cumbersome calculation in the section 1 can be simplified by calculating formally under the (non-projective) special Kähler condition, at least the symbols will be simpler, modulo some extra terms from non-projectivity. This is indeed the case, as is shown in Freed's paper [F].

We shall not repeat the calculation again in this notes, rather we would only briefly see the result and readers may refer to the calculation in [F]. It roughly states that there exists a correspondence between a cubic form $\mathcal{E} \in H^0(M, \text{Sym}^3 T^*M)$, which is the Yukawa coupling in our case (non-projectively), and the connection matrix A that gives the special Kähler connection ∇ via $\nabla = d + A$, where d is the Levi-Cevita connection.

In local coordinates, if we write $\omega = \sqrt{-1}b_{i\bar{j}}dz^i \wedge \overline{dz^j}$; $\mathcal{E} = \mathcal{E}_{ijk}dz^i \otimes dz^j \otimes dz^k$, then $(A_i)_{\bar{j}}^{\bar{k}} = \sqrt{-1}\mathcal{E}_{ijl}b^{l\bar{k}}$ while all other terms are 0, and the curvature tensor is given by

$$R_{i\bar{j}kl} = -b^{\alpha\bar{\beta}}\mathcal{E}_{ik\alpha}\overline{\mathcal{E}_{j\bar{l}\bar{\beta}}}.$$

One may find that the differences between the matrix here and that in our projective case only occur at indices 0 and $\bar{0}$, while in the curvature formula only the corresponding terms contribute difference. For instance, $e^{2\mathcal{K}}$ disappears here. These certainly looks neater than our original formulas. For actual coordinate-free calculations and statements of the correspondence, see [F].

3 Viewpoint of Bilagrangian

The rest of the survey will be devoted to an important feature of special Kähler manifolds that have been mentioned in previous parts. Namely, the special Kähler metric is totally determined by a single holomorphic function $\mathcal{F}$ in n variables. If we follow the calculation by local coordinates in section 1 of Freed's paper [F], we can obtain this result shortly. However, I choose to detour a little here to proceed as in Hitchin's paper [H] because I think essentially the calculations are analogue but we can see a little more structure via this method with better symmetry perceived.

We shall first recall the definition of a Lagrangian submanifold.

Def 3.1. A submanifold L of a symplectic manifold (M, ω) is said to be *Lagrangian* if for every point $P \in L$ and tangent vectors $X, Y \in T_P L$, we have $\omega|_P(X, Y) = 0$.

The idea of Hitchin is to focus more on the symplectic structure rather than the holomorphic structure, or to say, the complex structure. His main observation is the following:

Prop 3.1. A real $2n$ -dimensional submanifold $Y \subseteq X$ is complex Lagrangian if and only if $\omega^\epsilon = 0$, where (X, ω^ϵ) is a complex symplectic manifold of complex dimension $2n$ with the complex symplectic 2-form ω^ϵ .

For the simple but ingenious proof, the reader may refer to the paper [H]. However, the point is that if we separate the 2-form $\omega^\epsilon = \omega_1 + \sqrt{-1}\omega_2$, where ω_1, ω_2 are both real symplectic forms, the vanishing of ω^ϵ is simply determined by simultaneous vanishing of ω_1, ω_2 . We can give an argument that is similar to the proposition above in spirit, also to be used in further proofs, stating how these two real symplectic forms determine the structure of a complex manifold of X .

For this, consider the distribution $E \subseteq TX \otimes C$, where TX is the tangent bundle of X , which is defined by the complex vector fields U such that $\iota(U)\omega^\epsilon = 0$, where ι is the contraction. So this is just requiring $\omega^\epsilon(U, Y) \equiv 0$ for any complex vector field Y , i.e., $\omega_i(U, Y) \equiv 0$, $i = 1, 2$. If the algebraic condition $E \oplus \bar{E} = TX \otimes C$ holds, then we can define an almost complex structure on X by requiring $T^{0,1}X = E$ (since ω^ϵ is a $(2,0)$ -form, this is well-defined).

To show that this is indeed a complex structure, we only need to check integrability. By Cartan's magic formula, if $\iota(U)\omega^\epsilon = 0$, since ω^ϵ is closed by definition, we have

$$\mathcal{L}_U \omega^\epsilon = d(\iota(U)\omega^\epsilon) + \iota(U)d\omega^\epsilon = 0.$$

Hence if $U, V \in \Gamma(E)$, we have $\mathcal{L}_U \omega^\epsilon = 0$ and $\iota(V)\omega^\epsilon = 0$, and thus

$$\iota([U, V])\omega^\epsilon = \mathcal{L}_U(\iota(V)\omega^\epsilon) - \iota(V)(\mathcal{L}_U \omega^\epsilon) = 0,$$

showing that $[U, V] \in \Gamma(E)$ and thus implies the integrability.

With these preparations, next we shall show that to require a special Kähler structure is to some extent the same as to require a certain “bilagrangian” condition. We would see how the Kähler potential and the prepotential appear during the proof. The key observation is the following lemma.

Lem 3.1. Suppose we have a manifold M embedded in the space T^*V where V is a real symplectic vector space, and M is Lagrangian with respect to the canonical symplectic form ω on T^*V and transversal to the projection of $T^*V \cong V \times V^*$ to V . Then the embedding is defined by the graph of the derivative of a function ϕ on V , namely, in coordinates we have $y_j = \frac{\partial \phi}{\partial x_j}$, and moreover, if M is complex manifold, ϕ can be chosen to be holomorphic.

Proof. Reformulating the transversal condition, this is just saying that $\pi|_M : \rightarrow V$ is a local homeomorphism, where π is the canonical projection from T^*V to V . Moreover since M is embedded in T^*V and the latter is simply connected, this extends to a global homeomorphism. Hence M is globally parametrized by V , i.e., it is a graph for a section $\alpha : V \rightarrow V^*$ which is the same as an 1-form on V . Hence if we write $\alpha = \alpha_i(x)dx_i$ in coordinates of V with $x = (x_1, \dots, x_n)$, then $M = \{(x_1, \dots, x_n, \alpha_1(x), \dots, \alpha_n(x)) | x \in V\}$.

Next consider the Lagrangian condition. The canonical form ω on T^*V is $d\lambda$ where $\lambda = \sum y_j dx_j$ where y_i is a dual basis on V^* . Consider the parametrizing map $f : V \rightarrow M$ defined as above by $x \mapsto (x, \alpha(x))$. Then the Lagrangian condition $\omega|_M$ pulls back via f to be $f^*\omega = f^*d\lambda = df^*\lambda = 0$. But $f^*d\lambda = \sum_j (f^*y_j)f^*dx_j = \alpha_j(x)dx_j = d\alpha$. So the Lagrangian condition just says that $d\alpha = 0$ on V . Hence by Poincaré Lemma (where we can use it globally since V is just a vector space), we get $\alpha = d\phi$ for certain ϕ and hence $\alpha_j = \frac{\partial \phi}{\partial x_j}$, which is what we desired. $\square$

For a real vector symplectic vector space V with a skew form $\langle, \rangle$, we can define two forms Ω_1, Ω_2 and an indefinite metric g on $V \times V$ by

$$\begin{aligned}\Omega_1((a_1, a_2), (b_1, b_2)) &= \langle a_1, b_2 \rangle + \langle a_2, b_1 \rangle \\ \Omega_2((a_1, a_2), (b_1, b_2)) &= \langle a_1, b_2 \rangle - \langle a_2, b_1 \rangle \\ g((a, b), (a, b)) &= \frac{1}{2} \langle a, b \rangle\end{aligned}$$

Thm 3.1. Let $M \subseteq V \times V$ be a submanifold which is Lagrangian for Ω_1, Ω_2 and transversal to the two projections onto V . Then $g|_M$ is a special pseudo-Kähler metric. Conversely, any special pseudo-Kähler metric on a manifold M is locally induced from an embedding in $V \times V$ for certain V .

Proof. We can use the nondegenerate skew form $\langle, \rangle$ to identify V and V^* . Then by setting $d\xi_i = \sum \omega_{ij} dy_j$, Ω_1 becomes the canonical symplectic form on $T^*V \cong V \times V^*$ (up to a

scalar):

$$\begin{aligned}\Omega_1 &= 2 \sum dx_i \wedge d\xi_i; \\ \Omega_2 &= \sum \omega_{ij} dx_i \wedge dx_j + \sum \omega^{ij} d\xi_i \wedge d\xi_j.\end{aligned}$$

We can use the projection onto the first factor V to locally identify M with a flat symplectic space. In local coordinates $x_1, \dots, x_{2n}$, the ordinary derivative gives us our flat symplectic connection ∇ . Now consider M being a submanifold of $T^*V \cong V \times V$, by lemma 3.1, we find that the local coordinates is given by $\xi_j = \frac{\partial \phi}{\partial x_j}$ for a certain $\phi(x_1, \dots, x_{2n})$. Then by the chain rule, under the pushforward of this inclusion, the tangent vector $\frac{\partial}{\partial x_j}$ becomes

$$X_j = \frac{\partial}{\partial x_j} + \sum \frac{\partial \xi_l}{\partial x_j} \frac{\partial}{\partial \xi_l} = \frac{\partial}{\partial x_j} + \sum_l \frac{\partial^2 \phi}{\partial x_l \partial x_j} \frac{\partial}{\partial \xi_l}.$$

Recalling the definition of our metric, we see that the induced metric on M is

$$\begin{aligned}\sum g_{kj} dx_k dx_j &= \sum g((X_k, X_j), (X_k, X_j)) \\ &= \frac{1}{2} \left(\frac{\partial^2 \phi}{\partial x_k \partial x_j} - \left(-\frac{\partial^2 \phi}{\partial x_k \partial x_j} \right) \right) dx_k dx_j = \sum \frac{\partial^2 \phi}{\partial x_k \partial x_j} dx_k dx_j.\end{aligned}$$

It's direct to check this metric is non-degenerate by the transversality condition, though might not be positive in general.

Now we check the Lagrangian condition for Ω_2 . Then by the expressions of ξ_i, g_{kj} above, we get

$$0 = \Omega_2(X_k, X_j) = \omega_{kj} + \sum g_{ka} g_{jb} \omega^{ab}.$$

Thus if we take

$$J_k^j = \sum \omega^{ja} g_{ak},$$

we find that $J^2 = -1$ defining an almost complex structure. Also by the skew-symmetry of ω , we find that J is skew-adjoint with respect to g . Now take $a_j = \sum \omega^{ji} \frac{\partial \phi}{\partial x_i}$, we may find that $\frac{\partial a_j}{\partial x_k} = \sum \omega^{ji} g_{ik}$ since ω^{ji} is independent of x_k . Thus $J = d_{\nabla} X$ where $X = \sum a_j \frac{\partial}{\partial x_j}$. Hence $d_{\nabla} J = 0$ by the flatness of the connection.

It remains to show that J is integrable. Consider the complex functions $z_j = x_j - \sqrt{-1} \sum \omega^{jk} \frac{\partial \phi}{\partial x_k}$. Then we differentiate them to get $2n$ complex 1-forms:

$$dz_j = dx_j - \sqrt{-1} \sum \omega^{jk} g_{kl} dx_l = dx_j - \sqrt{-1} J_l^j dx_l = (\text{id} - \sqrt{-1} J) dx_j.$$

We see from the final expression that these are indeed $(1, 0)$ -forms. Moreover, since $2dx_j = dz_j + d\bar{z}_j$, if we consider the distribution E of the common kernel of $dz_1, \dots, dz_n$, we find that the complex rank of E is n and that $E \oplus \bar{E} = TM \otimes \mathbb{C}$. By a similar discussion to that after proposition 3.1, we find the almost complex structure is indeed integrable, and hence genuinely a complex struture. Thus the metric g , the flat symplectic connection ∇ and this complex structure J fits into a special pseudo-Kähler manifold.

Now let's prove the converse. Let M be a special pseudo-Kähler manifold and $(x_1, \dots, x_{2n})$ be flat local coordinates, so that the covariant derivative is just the ordinary derivative and the symplectic form is constant under this local coordinate. Then reversing the process above, if we define $\alpha_k = \sum \omega_{kl} J_j^l dx_j$, then since ω_{kl} is constant and $d_{\nabla} J = 0$, we have

$$d\alpha_k = \sum \omega_{kl} \frac{\partial J_j^l}{\partial x_m} dx_m \wedge dx_j = 0.$$

Thus locally there exists ξ_k such that $\alpha_k = d\xi_k$ and we can map M to $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ by $x \mapsto (x, \xi(x))$.

Then when restricting to M , by the symmetry of g_{jk} we have

$$\frac{1}{2}\Omega_1|_M = \sum dx_j \wedge \alpha_j = \sum dx_j \wedge \omega_{jl} J_k^l dx_k = \sum dx_j \wedge g_{jk} dx_k = 0.$$

And similarly by definition of α_j and computation we have

$$\Omega_2|_M = \sum \omega_{ij} dx_i \wedge dx_j + \omega^{ab} g_{ai} dx_i \wedge g_{bj} dx_j,$$

which can be seen to be 0 since $J^2 = -1$.

Then taking $V = \mathbb{R}^{2n}$ with the skew form given by ω_{ij} , we get a local embedding of M in the manner above, being Lagrangian with respect to Ω_1, Ω_2 with induced metric g_{ij} , hence we are done. $\square$

Finally, we shall see this ϕ appearing in the proof is indeed a Kähler potential (but in fact up to a scalar), as one may guess.

Prop 3.2. 2ϕ is a Kähler potential for the Kähler metric ω .

Proof.

$$\begin{aligned}
\sqrt{-1}\partial\bar{\partial}(2\phi) &= -\frac{1}{2}d(Jd(2\phi)) \\
&= -\left(\sum \frac{\partial}{\partial x_k}(J_j^i \frac{\partial \phi}{\partial x_i})dx_k \wedge dx_j\right) \\
&= -\left(\sum \omega^{ia} \frac{\partial^3 \phi}{\partial x_a \partial x_j \partial x_k} \frac{\partial \phi}{\partial x_i} dx_k \wedge dx_j + J_j^i \frac{\partial^2 \phi}{\partial x_i \partial x_k} dx_k \wedge dx_j\right).
\end{aligned}$$

The first term appears because of our construction of $J_j^i = \sum \omega^{ia} g_{aj}$ and the definition of $g_{aj} = \frac{\partial^2 \phi}{\partial x_a \partial x_j}$. Also note that the skew form ω is independent of x_k , hence implying the formula above. Also this first term vanishes due to the symmetry of i and j .

Finally we calculate that

$$\sum J_j^i g_{ik} dx_k \wedge dx_j = -\sum \omega_{kj} dx_k \wedge dx_j = -\omega,$$

and we are done. $\square$

Rmk 3.1. At the ending, we show how the prepotential also appears in this way. We choose now a symplectic basis on V and the corresponding local coordinates give $\omega = \sum dx_j \wedge dx_{n+j}$. In our notations above, we have now

$$\begin{aligned}
\Omega_1 &= 2 \sum_{j=1}^{2n} dx_j \wedge d\xi_j; \\
\Omega_2 &= \sum_{j=1}^n dx_j \wedge dx_{n+j} - \sum_{j=1}^n d\xi_j \wedge d\xi_{n+j}.
\end{aligned}$$

Then consider the complexified

$$\Omega^c = \frac{1}{2}\Omega_1 + \sqrt{-1}\Omega_2 = \sum_{j=1}^n d(x_j + \sqrt{-1}\xi_{n+j}) \wedge d(\xi_j + \sqrt{-1}x_{n+j}) = \sum_{j=1}^n dv_j \wedge dw_j,$$

hence being identified with the canonical complex symplectic form on $T^*\mathbb{C}^n$.

From proposition 3.1, a submanifold on which Ω_1, Ω_2 both vanish is the same as a submanifold of $T^*\mathbb{C}^n$, while from lemma 3.1, this is given by the graph of the derivative of a holomorphic function, i.e., $v_j = \frac{\partial \mathcal{F}}{\partial w_j}$. Thus as long as we have a special Kähler metric, we know from theorem 3.1 that this is locally induced from an embedding into $V \times V$ satisfying the transversality condition. Now using the theorem other way round, we see the special Kähler metric can always be induced by such a holomorphic prepotential $\mathcal{F}$.

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