

HAE in Holomorphic and Real Polarizations

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1 Introduction

Before we start, we can very briefly recall what we have discussed about till now. Roughly, we have seen 1) the idea of Witten to interpret the Holomorphic Anomaly Equation as the background independence of the quantum state $|\Psi_{\text{top}}\rangle$ associated to the topological string partition function Ψ_{top} via the geometric quantization process (using hol. polarization), and 2) how classically HAE is solved by Feynman rule expansion, through a heavy computation.

So these two aspects are kinda unrelated, or at least sitting in two worlds, one "pre-Witten", one "post-Witten". The point of this talk and the upcoming ones is partially to show how these two aspects are combined to pragmatically help us, at least on solving HAE's. But strictly speaking this will be dealt with only in the next time by the next speaker while this talk serves as some preparation.

The point of this talk are two-folds, one as preannounced last time, is the interpretation, or the following slogan: "HAE = variation of a wave function in complex polarization", while the other is to see how situations are different in real polarization, and determine the transformation of this wave function in the two polarizations.

Let's first recall one of the Dirac's rules required for geometric quantization, namely for a symplectic manifold (M, ω) and $f, g \in C^\infty(M)$, the quantized operator $\hat{f}$ shall satisfy the commutator relation $[\hat{f}, \hat{g}] = i\{\hat{f}, \hat{g}\}$, where we take $\hbar = 1$. In local coordinates if we write $\omega = \frac{1}{2}\omega_{ab}d\xi^a \wedge d\xi^b$, then the Poisson bracket is given by $\{f, g\} = \omega^{ab}\frac{\partial f}{\partial \xi^a}\frac{\partial g}{\partial \xi^b}$, where ω^{ab} is the inverse matrix (or exactly, that satisfies $\omega_{ac}\omega^{cb} = \delta_a^b$). Hence $\{\xi^a, \xi^b\} = \omega^{ab}$ and Dirac's rule yields that $[\hat{\xi}^a, \hat{\xi}^b] = i\omega^{ab}$ (also by the first two Dirac's rules). These would be basic and used later on.

2 Holomorphic Polarization

2.1 Basic notations and relations

We shall start with the holomorphic polarisation of $H^3(X, \mathbb{R})$, which comes from the familiar decomposition of $H^3(X, \mathbb{R}) \otimes \mathbb{C}$ into $H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$, with the corresponding decomposition of the dual γ of a 3-cycle C (the notation used in Verlinde's paper and also used last time), which goes as

$$\gamma = \frac{1}{2} \left(\lambda^{-1} \Omega + x^i \nabla_i \Omega + \overline{x^i \nabla_i \Omega} + \overline{\lambda^{-1} \Omega} \right).$$

Here as before $\nabla_i = \partial_i + \partial_i K$, where K is the Kähler potential, making ∇_i into a basis of $H^{2,1}$. The preferred λ^{-1} coordinate may come from the form of the attractor equations introduced last time, where we see a term $\lambda^{-1} X^I$. λ^{-1} and x^i varies with respect to t^i and $\bar{t}^{\bar{j}}$, inducing a family of different complex structures.

Now writing the natural symplectic pairing $\omega(\alpha, \beta) = \int_X \alpha \wedge \beta$ on $H^3(X, \mathbb{R})$ using the basis above, we get the form

$$\omega = i e^{-K} d\overline{\lambda^{-1}} \wedge d\lambda^{-1} + i e^{-K} G_{i\bar{j}} dx^i \wedge d\bar{x}^{\bar{j}},$$

which is deduced from the special geometry relations

$$\begin{aligned} -e^{-K} &= i \int_{\mathcal{M}} \overline{\Omega} \wedge \Omega, \\ -e^{-K} G_{i\bar{j}} &= i \int_{\mathcal{M}} \nabla_i \Omega \wedge \overline{\nabla_{\bar{j}} \Omega}. \end{aligned}$$

Now if we denote the corresponding operators after quantization by $\widehat{x^i}, \widehat{\lambda^{-1}}$, they shall satisfy the commutator relations which we have seen in the end of §1:

$$\begin{aligned} [\widehat{\lambda^{-1}}, \widehat{\lambda^{-1}}] &= e^K, \\ [\widehat{x^i}, \widehat{\bar{x}^{\bar{j}}}] &= e^K G^{i\bar{j}}, \end{aligned}$$

while all other commutator relations are trivial. In quantum theory, one usually call a pair of operators $\widehat{a}, \widehat{a}^\dagger$ that satisfies the commutator relation $[\widehat{a}, \widehat{a}^\dagger] = 1$ as a pair of annihilation and creation operator, respectively. So after normalizing the equations above, we can say that $\widehat{x^i}$ and $\widehat{\lambda^{-1}}$ behaves as annihilation operators, while $\widehat{\bar{x}^{\bar{j}}}$ and $\widehat{\lambda^{-1}}$ as creation operators. We have the following result concerning the eigenstate of an annihilation operator.

Prop 2.1. For an annihilation operator (or a family of them that pairwise commute), the point spectrum is entire $\mathbb{C}$, and for every eigenvalue α , there exists a unique normalized eigenstate $|\alpha\rangle$, called the coherent state, and these states form a basis of the whole space.

In fact, one can directly construct an eigenvector for $\alpha \in \mathbb{C}$, namely $|\alpha\rangle = \exp(\alpha \widehat{a}^\dagger - \alpha^* \widehat{a})|0\rangle$, where $|0\rangle$ is the vacuum state. For uniqueness, one may refer to literatures concerning Stone-von Neumann Thm. and further implications.

Now we can speak of the coherent eigenstates $|x, \lambda\rangle$ of $\widehat{\lambda}^{-1}$ and (all) $\widehat{x}^i$'s, since they all commute, namely

$$\widehat{x}^i |x, \lambda\rangle = x^i |x, \lambda\rangle \quad (1)$$

$$\widehat{\lambda}^{-1} |x, \lambda\rangle = \lambda^{-1} |x, \lambda\rangle. \quad (2)$$

But there is a problem that $\widehat{\lambda}^{-1}$ behaves as creation operator instead of annihilation operator. We'd just pretend it to be for now and formally fix this later.



Before we proceed, let me explain how the quantities vary in these two equations, which is very important to avoid confusion in the later computation. Here $\widehat{x}^i$ is the quantized operator of the coordinate function x^i , while x^i on the right hand side indicates a certain eigenvalue of $\widehat{x}^i$, which takes every possible value in $\mathbb{C}$ and is independent from $t^i, \bar{t}^i$! So x^i does not vary as background $t^i, \bar{t}^i$ vary, we would pick a preferred complex number x^i in equation (1) and let itself vary, while $\widehat{x}^i$ certainly varies with $t^i, \bar{t}^i$ but is independent from x^i . In a word, we have the relations

$$\begin{aligned} \bar{\partial}_{\bar{t}^j} x^i &= 0, \quad \forall i, j \\ \frac{\partial}{\partial x^j} \widehat{x}^i &= 0, \quad \forall i, j. \end{aligned} \quad (3)$$

2.2 Topological string partition function

Now let's turn to the thing that we originally care about: Ψ_{top} , which in Verlinde's paper is defined to be

$$\begin{aligned} \Psi_{\text{top}}(x, \lambda; t, \bar{t}) &= \lambda^{\frac{\chi}{24}-1} \exp\left(\sum_g \lambda^{2g-2} W_g(\lambda x; t, \bar{t})\right), \\ &= \lambda^{\frac{\chi}{24}-1} \exp\left(\sum_g \lambda^{2g-2} \left(\sum_{n=0}^{\infty} \frac{1}{n!} C_{i_1 \dots i_n}^{(g)} (\lambda x)^{i_1} \dots (\lambda x)^{i_n}\right)\right). \end{aligned}$$

Then

$$\log \Psi_{\text{top}} = \sum_{g,n} \frac{1}{n!} \lambda^{2g-2+n} C_{i_1 \dots i_n}^{(g)} x^{i_1} \dots x^{i_n} + \left(\frac{\chi}{24} - 1\right) \log \lambda.$$

Now we should find the correct correspondence of this Ψ_{top} in BCOV paper: recall that we have

$$\tilde{W}(\lambda_B, x, \varphi; t, \bar{t}) = \sum_{g,n} \frac{1}{n!} \lambda_B^{2g-2} C_{i_1 \dots i_n}^{(g)} x^{i_1} \dots x^{i_n} (1 - \varphi)^{-(2g-2+n)} - \left(\frac{\chi}{24} - 1\right) \log(1 - \varphi).$$

So the correct way to understand Verlinde's Ψ_{top} is $\Psi_{\text{top}}(x, \lambda; t, \bar{t}) = \tilde{W}(1, x, 1 - \lambda^{-1}; t, \bar{t})$.

Now recall that the form of HAE we derived for $\tilde{W}$ is

$$\bar{\partial}_{\bar{i}} e^{\tilde{W}} = \left(\frac{\lambda_B^2}{2} e^{2K} G^{j\bar{m}} G^{k\bar{n}} \bar{C}_{i\bar{m}\bar{n}} \frac{\partial^2}{\partial x^j \partial x^k} - G_{ij} x^j \frac{\partial}{\partial \varphi} \right) e^{\tilde{W}}.$$

Now take $\lambda_B = 1$, and since in Verlinde's convention we have $\varphi = 1 - \lambda^{-1}$, $G^{j\bar{m}} G^{k\bar{n}} \bar{C}_{i\bar{m}\bar{n}} = \bar{C}_{\bar{i}}^{jk}$, we have our desired first of the HAE:

$$\bar{\partial}_{\bar{i}} \Psi_{\text{top}} = \left[\frac{e^{2K}}{2} \bar{C}_{\bar{i}}^{jk} \frac{\partial^2}{\partial x^j \partial x^k} + G_{ij} x^j \frac{\partial}{\partial \lambda^{-1}} \right] \Psi_{\text{top}}.$$

The other half of the HAE reads

$$[\nabla_i + I_{ij}^k x^j \frac{\partial}{\partial x^k}] \Psi_{\text{top}} = \left[\lambda^{-1} \frac{\partial}{\partial x^i} - \frac{1}{2} \partial_i \log |G| - \frac{1}{2} C_{ijk} x^j x^k \right] \Psi_{\text{top}}.$$

Now it's time to summarize Verlinde's idea: we can form a state in the Hilbert space $\mathcal{H}_{H^3(X, \mathbb{R})}$ using the basis given by the coherent states, such that $\Psi_{\text{top}}(x, \lambda; t, \bar{t}) = \langle \Psi_{\text{top}} | x, \lambda \rangle$. This is simply obtained by the expansion

$$|\Psi_{\text{top}}\rangle = \int dx d\lambda^{-1} \Psi_{\text{top}}(x, \lambda) |x, \lambda\rangle.$$

We shall prove that $|\Psi_{\text{top}}\rangle$ is a background independent state, making $\Psi_{\text{top}}(x, \lambda; t, \bar{t})$ into a wave function of this state, so that we can say that HAE is the variation of wave function under different complex polarizations.

Note that this is indeed proving that the background dependence of the wave function is all contained and induced by that of the coherent states $|x, \lambda\rangle$, so our calculation is indeed calculating similar HAE equations for $|x, \lambda\rangle$. For instance, to show $\bar{\partial}_{\bar{i}} |\Psi_{\text{top}}\rangle = 0$, HAE for Ψ_{top} reduces to show

$$0 = \int dx d\lambda^{-1} \left[\left(\frac{e^{2K}}{2} \bar{C}_{\bar{i}}^{jk} \frac{\partial^2}{\partial x^j \partial x^k} + G_{ij} x^j \frac{\partial}{\partial \lambda^{-1}} \right) \Psi_{\text{top}} + \Psi_{\text{top}} \bar{\partial}_{\bar{i}} \right] |x, \lambda\rangle.$$

Integration by parts we see this implies

$$0 = \int dx d\lambda^{-1} \Psi_{\text{top}} \left(\bar{\partial}_{\bar{i}} - G_{ij} x^j \frac{\partial}{\partial \lambda^{-1}} + \frac{e^{2K}}{2} \bar{C}_{\bar{i}}^{jk} \frac{\partial^2}{\partial x^j \partial x^k} \right) |x, \lambda\rangle.$$

So in fact we are remained to show the following for $|x, \lambda\rangle$:

$$\bar{\partial}_{\bar{i}} |x, \lambda\rangle = -\frac{e^{2K}}{2} \bar{C}_{\bar{i}}^{jk} \frac{\partial^2}{\partial x^j \partial x^k} |x, \lambda\rangle + G_{ij} x^j \frac{\partial}{\partial \lambda^{-1}} |x, \lambda\rangle.$$

Thus it is perfectly reasonable to believe this is a calculation boiled down to the special geometry relations, as is indeed so. We'll carry out the computation in next section.

2.3 Calculations of HAE

Now let operators $\widehat{x^i}, \widehat{\lambda^{-1}}$ vary as $t^i, \bar{t}^{\bar{i}}$ vary. The goal is to show that $|\Psi_{\text{top}}\rangle$ itself is independent from t^i and $\bar{t}^{\bar{i}}$ via the computation on HAE that the coherent states $|x, \lambda\rangle$ satisfy, which indeed gives our desired result by the previous discussion.

First we need to deduce the variation of the operators $\widehat{\lambda^{-1}}$ and $\widehat{x^k}$ w.r.t. $t^i, \bar{t}^{\bar{j}}$. The constraints is due to that the chosen γ is topological and independent of background $t^i, \bar{t}^{\bar{j}}$. So for example from $\bar{\partial}_{\bar{i}}\gamma = 0$ we get

$$\bar{\partial}_{\bar{i}}\widehat{x^k} = e^K \bar{C}_{ij}^k \widehat{\bar{x}^{\bar{j}}} \quad (4)$$

$$\bar{\partial}_{\bar{i}}\widehat{\lambda^{-1}} = -G_{ij} \widehat{\bar{x}^{\bar{j}}}, \quad (5)$$

using the special geometry relations (mod exact terms)

$$\bar{\partial}_{\bar{i}}\nabla_j\Omega = G_{ij}\Omega, \quad D_i\bar{\nabla}_j\Omega = -e^K C_{ij}^k \bar{\nabla}_{\bar{k}}\bar{\Omega}.$$

Now we are ready to verify that the coherent states satisfy the following form of holomorphic anomaly:

$$\bar{\partial}_{\bar{i}}|x, \lambda\rangle = \left(-\frac{e^{2K}}{2} \bar{C}_{\bar{i}}^{jk} \frac{\partial^2}{\partial x^j \partial x^k} + G_{ij} x^j \frac{\partial}{\partial \lambda^{-1}} \right) |x, \lambda\rangle \quad (6)$$

Proof. We start from equations (1) and (2), act respectively on these equations by $\bar{\partial}_{\bar{i}}, \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^m}$, using equations (3) and other similar independence relations, by Leibniz rule we obtain:

$$(\widehat{x^k} - x^k) \bar{\partial}_{\bar{i}}|x, \lambda\rangle = -(\bar{\partial}_{\bar{i}}\widehat{x^k})|x, \lambda\rangle \quad (7)$$

$$(\widehat{\lambda^{-1}} - \lambda^{-1}) \bar{\partial}_{\bar{i}}|x, \lambda\rangle = -(\bar{\partial}_{\bar{i}}\widehat{\lambda^{-1}})|x, \lambda\rangle \quad (8)$$

$$(\widehat{x^k} - x^k) \frac{\partial}{\partial x^l}|x, \lambda\rangle = \delta_l^k |x, \lambda\rangle \quad (9)$$

$$(\widehat{\lambda^{-1}} - \lambda^{-1}) \frac{\partial}{\partial \lambda^{-1}}|x, \lambda\rangle = |x, \lambda\rangle \quad (10)$$

Now we substitute the equations (4), (5) into (7), (8) respectively to obtain

$$(\widehat{x^k} - x^k) \bar{\partial}_{\bar{i}}|x, \lambda\rangle = -e^K \bar{C}_{ij}^k \widehat{\bar{x}^{\bar{j}}}|x, \lambda\rangle \quad (11)$$

$$(\widehat{\lambda^{-1}} - \lambda^{-1}) \bar{\partial}_{\bar{i}}|x, \lambda\rangle = G_{ij} \widehat{\bar{x}^{\bar{j}}}|x, \lambda\rangle = G_{ij} x^j |x, \lambda\rangle. \quad (12)$$

In fact compare (12) and (10), we have already get the second half of the operator $\bar{\partial}_{\bar{i}}$ as in (6). So now we are remained to compare (11) to (9), and for this purpose we need to know how $\widehat{\bar{x}^{\bar{j}}}$ acts on the state $|x, \lambda\rangle$.

Now observe that

$$(\widehat{x^k} - x^k) \widehat{\bar{x}^{\bar{j}}}|x, \lambda\rangle = (\widehat{x^k \bar{x}^{\bar{j}}} - \widehat{\bar{x}^{\bar{j}} x^k})|x, \lambda\rangle = [\widehat{x^k}, \widehat{\bar{x}^{\bar{j}}}]|x, \lambda\rangle = e^K G^{k\bar{j}}|x, \lambda\rangle \quad (13)$$

Compare (13) and (9), we deduce that

$$\widehat{\bar{x}^{\bar{j}}}|x, \lambda\rangle = e^K G^{l\bar{j}} \frac{\partial}{\partial x^l}|x, \lambda\rangle \quad (14)$$

Substituting (14) into (11) and we get

$$(\widehat{x^k} - x^k) \bar{\partial}_{\bar{i}}|x, \lambda\rangle = -e^{2K} \bar{C}_{ij}^k G^{l\bar{j}} \frac{\partial}{\partial x^l}|x, \lambda\rangle = -e^{2K} \bar{C}_{\bar{i}}^{kl} \frac{\partial}{\partial x^l}|x, \lambda\rangle \quad (15)$$

Now if we take again the derivative $\frac{\partial}{\partial x^m}$ of (9), we get

$$(\widehat{x^k} - x^k) \frac{\partial^2}{\partial x^l \partial x^m} |x, \lambda\rangle = \left(\delta_m^k \frac{\partial}{\partial x^l} + \delta_l^k \frac{\partial}{\partial x^m} \right) |x, \lambda\rangle \quad (16)$$

which implies further that

$$(\widehat{x^k} - x^k) \cdot \frac{1}{2} e^{2K} \overline{C_i}^{lm} \frac{\partial^2}{\partial x^l \partial x^m} |x, \lambda\rangle = e^{2K} \overline{C_i}^{kl} \frac{\partial}{\partial x^l} |x, \lambda\rangle \quad (17)$$

Comparing (17) and (15), we finally get the first part of $\bar{\partial}_i$, hence add these two parts together we have proved (6). $\square$

2.4 Remarks

In this section I shall give some remarks, which are stuffs that I myself has not checked or isn't familiar with.

Rmk 2.1. Recall there is another HAE equation, concerning the action of $\nabla_i + I_{ij}^k x^j \frac{\partial}{\partial x^k}$ acting on Ψ_{top} . Through an exactly same process, one could pass from this equation to HAE equation for $|x, \lambda\rangle$, with action by $\nabla_i + I_{ij}^k x^j \frac{\partial}{\partial x^k}$ instead of $\bar{\partial}_i$. And in the calculation process, the basic relations (7)-(10) still hold, so what should be changed is merely special geometry relations (4) & (5), where we'd use that for $\nabla_i + I_{ij}^k x^j \frac{\partial}{\partial x^k}$. Mutatis Mutandis, this finishes the other half to check $|\Psi_{\text{top}}\rangle$ is independent from t^i moduli, checking its background independence.

There is another equivalent way to proceed, by noticing that the second HAE for Ψ_{top} is the conjugate of the first w.r.t. the following background independent norm:

$$\langle \Psi_{\text{top}} | \Psi_{\text{top}} \rangle = \int d\mu_{x,\lambda} \exp \left[-e^{-K} G_{ij} x^i \bar{x}^j + e^{-K} (\bar{\lambda}^{-1} \lambda^{-1}) \right] |\Psi_{\text{top}}|^2,$$

where $d\mu_{x,\lambda} = d^2 \lambda d^{2b} x |\lambda|^{-4} e^{\frac{b+1}{2} K} |G|^{\frac{1}{4}}$. I'm too tired to carry out the calculation for this and I just include this for sake of completeness.

Rmk 2.2. Recall that we have a problem that $\widehat{\lambda}^{-1}$ behaves like a creation instead of annihilation operator, so $|x, \lambda\rangle$ do not really live in the Hilbert space, and this also leads to the term $e^{-K} (\bar{\lambda}^{-1} \lambda^{-1})$ in the integral, which cause the formal integral to be divergent. A way to solve this is by introducing the central charge $\widehat{Q}$ as defined before, which satisfies $\widehat{\lambda}^{-1} = e^K \widehat{Q}$ and thus we have $[\widehat{Q}, \widehat{Q}] = e^{-K}$, and now $\widehat{Q}$ is indeed annihilation operator, and we formally write

$$|x, \lambda\rangle = \int dQ \exp(-\lambda^{-1} \widehat{Q}) |x, Q\rangle.$$

Rmk 2.3. I think Verlinde in his paper defines $|\Psi_{\text{top}}\rangle$ as a state first using the basis given by $|x, \lambda\rangle$ and prove its own background independence. I'm not quite sure whether there is a way to define the state $|\Psi_{\text{top}}\rangle$ itself intrinsically through the quantization process, and reversely prove the HAE for Ψ_{top} as a wave function defined by inner product $\langle \Psi_{\text{top}} | x, \lambda \rangle$.

3 Real polarization and the relation between polarizations.

Finally we would take some time to have a glimpse at real polarizations. This is obtained by considering the symplectic form $\omega = \sum_I dp^I \wedge dq_I$ where p^I, q_I are real, defined topologically by $\int_{A^I} \omega = p^I, \int_{B_I} \omega = q_I$, where A^I, B_I are 3-cycles s.t. $A^I \cap B_I = \partial_J^I$.

Again by requirement of quantization, we find that $[\widehat{p^I}, \widehat{q_I}] = -i\delta_J^I$, and $\widehat{p^I}, \widehat{q_I}$ are Hermitian operators since p^I, q_I are real. Either the eigenstates of $\widehat{p^I}$ or of $\widehat{q_I}$ span the Hilbert space and the convention here is to use that of $\widehat{p^I}$. Thus we can likewise discuss about the eigenstates $|p\rangle$ and the wave function $\Psi_{\text{top}}(p) = \langle \Psi_{\text{top}} | p \rangle$ in terms of these. And they form a complete basis of the Hilbert space, namely

$$1 = \int dp |p\rangle \langle p|.$$

In Verlinde's paper he briefly mentioned the monodromy property of the eigenstate $|p\rangle$, and of $\Psi_{\text{top}}(p)$, due to the choice of 3-cycles A^I, B_I at the first stage. But I think this would be discussed in much more details in later talks so I just skip it for now since it may seem a little isolated. Rather, I would turn to another natural question, that how are these two polarizations related, especially how the wave function transforms.

For this purpose it would be convenient to use a combined notation, namely $x^I = \lambda^{-1} X^I + x^i \nabla_i X^I$ where X^I is the projective coordinate, also introduced in the lecture last time. (In physics these variables x^I are said to live in the "big phase space") Now we have the base change relations

$$\text{Re}(x^I) = p^I, \quad \text{Re}(\tau_{IJ} x^J) = q_I \quad (18)$$

where $\tau_{IJ} = \frac{\partial F_I}{\partial x^J}$ and F_I are also introduced last time. These relations are also a mere rewriting of that mentioned last time:

$$\text{Re}(\lambda^{-1} X^I + x^i \nabla_i X^I) = p^I, \quad \text{Re}(\lambda^{-1} F_I + x^i \nabla_i F_I) = q_I.$$

Now from this relation (18) one also solves that $x^I = -i(\text{Im}\tau_{IJ})^{-1}(q_J - \bar{\tau}_{JK} p^K)$.

The previous coherent states $|x, \lambda\rangle$ can be identified with the eigenstates $|x\rangle$ of $\widehat{x^I}$ by comparing commutator relations. To deduce $\Psi_{\text{top}}(x)$ in terms of $\Psi_{\text{top}}(p)$, using completeness relation, we want to calculate

$$\langle \Psi_{\text{top}} | x \rangle = \langle \Psi_{\text{top}} | 1 | x \rangle = \langle \Psi_{\text{top}} | \int dp |p\rangle \langle p | x \rangle = \int dp \Psi_{\text{top}}(p) \langle p | x \rangle.$$

We are remained to calculate the base change transform $\langle p | x \rangle$, using the relations (18) above. We first upgrade it to a version for operators after quantization namely,

$$\text{Re}(\widehat{x^I}) = \widehat{p^I}, \quad \text{Re}(\tau_{IJ} \widehat{x^J}) = \widehat{q_I} \quad (19)$$

Then we know that

$$\widehat{p^I} = \frac{1}{2}(\widehat{x^I} + \widehat{\bar{x}^I}), \quad \widehat{q_I} = \frac{1}{2}(\tau_{IJ} \widehat{x^J} + \bar{\tau}_{IJ} \widehat{\bar{x}^J}) \quad (20)$$

leading together to

$$\widehat{q}_I = \frac{1}{2}(\tau_{IJ}\widehat{x}^J + \bar{\tau}_{IJ}(2\widehat{p}^J - \widehat{x}^J)) = \bar{\tau}_{IJ}\widehat{p}^J + i(\text{Im}\tau_{IJ})\widehat{x}^J \quad (21)$$

Now since $|p\rangle, |x\rangle$ are eigenstates of $\widehat{p}^J, \widehat{x}^J$ respectively, we know that

$$\langle p|\widehat{q}_I|x\rangle = \langle p|(\bar{\tau}_{IJ}\widehat{p}^J + i(\text{Im}\tau_{IJ})\widehat{x}^J)|x\rangle = (\bar{\tau}_{IJ}p^J + i(\text{Im}\tau_{IJ})x^J)\langle p|x\rangle. \quad (22)$$

Similar to our previous discussion on how $\widehat{x}^J$ acts on the coherent states $|x\rangle$ (equation (14)), we can also deduce how $\widehat{q}_I$ acts on the (dual of) eigenstates $\langle p|$ from commutator relations, that is:

$$\langle p|\widehat{q}_I = -i\frac{\partial}{\partial p^I}\langle p| \quad (23)$$

Now we have a first order differential equation for $\langle p|x\rangle$:

$$-i\frac{\partial}{\partial p^I}\langle p|x\rangle = (\bar{\tau}_{IJ}p^J + i(\text{Im}\tau_{IJ})x^J)\langle p|x\rangle. \quad (24)$$

Now we denote $x_I := \text{Im}\tau_{IJ}x^J$, and rewrite it to be

$$\frac{-i}{\langle p|x\rangle}\frac{\partial\langle p|x\rangle}{\partial p^I} = (i\bar{\tau}_{IJ}p^J - (\text{Im}\tau_{IJ})x^J) \quad (25)$$

It is clear now we have a solution

$$\langle p|x\rangle = \exp\left(\frac{i}{2}(p^I\bar{\tau}_{IJ}p^J + 2ip^I(\text{Im}\tau_{IJ})x^J) + C\right), \quad (26)$$

where C is a constant depending on x , by requiring some normalizing properties. For an explicit choice, see the equation (5.19) & (5.20) in Verlinde's paper.